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A general scheme for studying the stochastic dynamics of a parametric oscillator driven by coloured noise

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Abstract

The stochastic dynamics of a classical parametric oscillator driven by coloured noise with exponential correlation is studied both by analytical and numerical methods. The critical strength of the parametric modulation at which the motion of the oscillator becomes unbounded, and the dependence of this critical strength on the magnitude and the correlation length of the noise, are investigated. The results show that the critical strength does not depend on either the strength or the correlation length of the coloured noise, but that the squeezing induced by the parametric oscillation is enhanced by increasing the correlation length of the noise.

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1. Introduction

Various studies of stochastic dynamical systems driven by coloured noise have been carried out. In previous works on the effects of coloured noise, several methods have been developed for characterizing stationary processes described by first-order nonlinear stochastic differential equations, including the unified coloured noise approximation [1], continued fraction methods [2] and the path integral method [3]. However, little work has been done on non-stationary stochastic processes, except for the case of parametric oscillators driven by multiplicative coloured noise [4–6]. Recently, Zerbe *et al* [7] studied the stochastic dynamics of dissipative Floquet oscillators driven by Gaussian white noise (GWN). This analytically solvable non-stationary system exhibits the squeezing of noise induced by the frequency modulation in a Brownian parametric oscillator, which has potential applications in electromagnetic interferometers used for noise reduction in communication networks. In fact, studies based on a stability analysis of the Floquet spectrum have shown that noise reduction can be achieved by redistributing the thermal noise [8–10] and that this redistribution can be realized by modulating the frequency of the Brownian oscillator [7, 11].

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The noise source describes the heat bath of the irrelevant variables which are assumed to have relaxed to equilibrium, independent of the dynamics of the system variables. As can be seen from the generalized Langevin equation, the fluctuations around the system induced by the surrounding heat bath have a correlation of finite time due to a built in delay or inertial effects [12]. In the present paper, we focus on the effect of coloured noise with an exponential autocorrelation. This type of noise is a second-order approximation from the viewpoint of the fluctuation–dissipation theorem [13]. Several previous works including Leiber *et al* have shown that the stationary distribution deviates from the Maxwellian for the system driven by coloured noise [14–16]. Our purpose is to extend our understanding of the effect of external coloured noise to the system governed by a non-stationary stochastic process. In section 2, we present the stochastic model driven by coloured noise and formally derive the probability distribution of a non-stationary Fokker–Planck process based on the Green function approach. In section 3, the equations of motion of each element of the covariance matrix are derived and the time-dependent behaviour of the coordinate variance σ_{xx} is studied by applying three different numerical techniques. In section 4, the effect of coloured noise on the squeezing induced by a parametric oscillation is studied both by the perturbation technique and by the full numerical method. Section 5 examines the instability of the oscillator by calculating the Floquet spectrum of the corresponding Fokker–Planck operator, and compares this system with that of the system driven by GWN. Our conclusions are presented in section 6.

2. Model and solutions of basic equations

Let us start with the following equation describing the stochastic dynamics of a dissipative Floquet oscillator driven by coloured noise:

$$\ddot{x} + \gamma\dot{x} + \left[\omega_0^2 + 2\epsilon\cos(2t+\psi)\right]x = c(t). \tag{1}$$

Here ϵ is the amplitude of the parametric modulation, γ and ω_0 are the damping constant and the natural frequency of the harmonic potential, respectively, and $c(t)$ is a random external perturbation. In the absence of noise, equation (1) can be transformed to the standard form of the Mathieu equation by substituting $x = \mathrm{e}^{-\gamma t/2}y$.

$$\ddot{y} + \left[\omega_0^2 - \frac{\gamma^2}{4} + 2\epsilon\cos(2t+\psi)\right]y = 0. \tag{2}$$

The Mathieu equation cannot be solved analytically. However, Floquet theory tells us that the solution of equation (2) can be expressed as a linear combination of two periodic functions multiplied by exponential cofactors.

$$\begin{aligned} y_1(t) &= \exp(\mathrm{i}\nu(t+\psi/2))\,p\left(t+\frac{\psi}{2}\right)\\ y_2(t) &= \exp(-\mathrm{i}\nu(t+\psi/2))\,p\left(-t-\frac{\psi}{2}\right) \end{aligned} \tag{3}$$

where ν is the Mathieu characteristic exponent and p is a π-periodic function. Then the solution of equation (1) in homogeneous form (i.e., in the absence of noise) $x_h(t)$ is as follows:

$$\begin{aligned} x_h(t) = \beta_1\ &\exp([\mathrm{i}\nu - (\gamma/2)]t + \mathrm{i}\nu(\psi/2))\,p\left(t+\frac{\psi}{2}\right)\\ &+\beta_2\ \exp(-[\mathrm{i}\nu + (\gamma/2)]t - \mathrm{i}\nu(\psi/2))\,p\left(-t-\frac{\psi}{2}\right). \end{aligned} \tag{4}$$

The above solution helps us to construct the formal solution of equation (1) by applying the Green function expression of the following form:

$$x(t) = x_h(t) + \int_0^t G_x(t, t')c(t')\,\mathrm{d}t'. \tag{5}$$

Here $G_x(t, t')$ is defined as

$$G_x(t, t') = [\phi_1(t)\phi_2(t') - \phi_1(t')\phi_2(t)]\ \exp(-\gamma(t - t')/2) \tag{6}$$

where ϕ_1 and ϕ_2 are two independent solutions of equation (2) satisfying the initial conditions $\phi_1(0) = 0$, $\phi_2(0) = 1$, $\dot{\phi}_1(0) = 1$ and $\dot{\phi}_2(0) = 0$.

As is well known, when $c(t)$ is a GWN $\Gamma(t)$ satisfying $\langle\Gamma(t)\Gamma(t')\rangle = 2D\gamma\delta(t - t')$, the probability distribution of an integrated dynamical quantity defined as $R = \int_0^t f(t, t')\Gamma(t')\,\mathrm{d}t'$ has a Gaussian distribution with variance $\sigma^2 = 2D\gamma \int_0^t f^2(t, t')\,\mathrm{d}t'$. Extending the present formalism to the case of coloured noise is straightforward. The external perturbation $c(t)$ is assumed to be Gaussian coloured noise with zero mean and exponential-type correlation.

$$\langle c(t)c(t')\rangle = \frac{D\gamma}{\tau}\ \exp(-|t - t'|/\tau). \tag{7}$$

Then the motion of $c(t)$ is governed by

$$\dot{c} = -\frac{1}{\tau}c + \frac{1}{\tau}\Gamma(t). \tag{8}$$

Solving equation (8),

$$c(t) = \langle c(t)\rangle + \frac{1}{\tau}\int_0^t G_c(t, t')\Gamma(t')\,\mathrm{d}t' \tag{9}$$

where $\langle c(t)\rangle = c_0 \exp(-t/\tau)$ and $G_c(t, t') = \exp(-|t - t'|/\tau)$. Substituting equation (9) into equation (5) and rearranging the order of integration, we obtain the following formal solution of $x(t)$:

$$x(t) = \langle x(t)\rangle + \frac{1}{\tau}\int_0^t \left[\int_{t''}^t G_x(t, t')G_c(t', t'')\,\mathrm{d}t'\right]\Gamma(t'')\,\mathrm{d}t'' \tag{10}$$

where $\langle x(t)\rangle = x_h(t) + \int_0^t \langle c(t)\rangle G_x(t, t')\,\mathrm{d}t'$. In addition, we obtain the formal solution of $v(t) = \dot{x}(t)$

$$v(t) = \langle v(t)\rangle + \frac{1}{\tau}\int_0^t \left[\int_{t''}^t \frac{\mathrm{d}}{\mathrm{d}t}G_x(t, t')G_c(t', t'')\,\mathrm{d}t'\right]\Gamma(t'')\,\mathrm{d}t'' \tag{11}$$

where $\langle v(t)\rangle = x_h{}'(t) + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \langle c(t)\rangle G_x(t, t')\,\mathrm{d}t'$. σ_{xx}, σ_{xv} and σ_{vv} can be obtained from equations (10) and (11) (see equation (17)). Then the probability distribution function $W(x, v, t)$ can be written as

$$W(x, v, t) = \frac{1}{2\pi\sqrt{\sigma_{xx}\sigma_{vv} - \sigma_{xv}^2}}\exp\left[-\frac{z}{2\left(\sigma_{xx}\sigma_{vv} - \sigma_{xv}^2\right)}\right] \tag{12}$$

where $z = \sigma_{vv}(x - \langle x\rangle)^2 - 2\sigma_{xv}(x - \langle x\rangle)(v - \langle v\rangle) + \sigma_{xx}(v - \langle v\rangle)^2$. The evolution of the process described by the variable (x, v) explicitly depends on the initial value of the noise variable, indicating that the stochastic process is non-Markovian. This non-Markovian nature can be easily eliminated by extending the dynamical variables (x, v) to (x, v, c). Then the probability distribution function $W(x, v, c, t)$ satisfies the following form of the Fokker–Planck equation:

$$\frac{\partial}{\partial t}W(x, v, c, t) = \mathcal{L}_{\mathrm{FP}}W(x, v, c, t) \tag{13}$$

where

$$\mathcal{L}_{\mathrm{FP}} = -\frac{\partial}{\partial x}v + \alpha(t)\frac{\partial}{\partial v}x + \gamma\frac{\partial}{\partial v}v - \frac{\partial}{\partial v}c + \frac{1}{\tau}\frac{\partial}{\partial c}c + \frac{D\gamma}{\tau^2}\frac{\partial^2}{\partial c^2} \tag{14}$$

and $\alpha(t)$ being $\omega_0^2 + 2\epsilon\cos(2t+\psi)$. The Fokker–Planck process in (x, v, c) can be uniquely described by a trivariate normal probability distribution.

$$W(\mathrm{x}, t) = \frac{1}{\sqrt{(2\pi)^3\det\sigma}}\exp\left[-\frac{1}{2}(\mathrm{x}-\bar{\mathrm{x}})^T\sigma^{-1}(\mathrm{x}-\bar{\mathrm{x}})\right] \tag{15}$$

where $\mathrm{x} = (x, v, c)$, $\bar{\mathrm{x}}(t) = (\langle x(t)\rangle, \langle v(t)\rangle, \langle c(t)\rangle)$ and

$$\sigma(t) = \begin{bmatrix} \sigma_{xx}(t) & \sigma_{xv}(t) & \sigma_{xc}(t) \\ \sigma_{xv}(t) & \sigma_{vv}(t) & \sigma_{vc}(t) \\ \sigma_{xc}(t) & \sigma_{vc}(t) & \sigma_{cc}(t) \end{bmatrix}. \tag{16}$$

From equations (9), (10) and (11), the elements of the covariance matrix $\sigma(t)$ can be obtained as follows:

$$\begin{aligned}
\sigma_{xx} &= \frac{2D\gamma}{\tau^2}\int_0^t \left[\int_{t''}^t G_x(t,t')G_c(t',t'')\,\mathrm{d}t'\right]^2 \mathrm{d}t'' \\
\sigma_{xv} &= \frac{2D\gamma}{\tau^2}\int_0^t \left[\int_{t''}^t G_x(t,t')G_c(t',t'')\,\mathrm{d}t' \cdot \int_{t''}^t \frac{\partial}{\partial t}G_x(t,t')G_c(t',t'')\,\mathrm{d}t'\right] \mathrm{d}t'' \\
\sigma_{xc} &= \frac{2D\gamma}{\tau^2}\int_0^t \left[\int_{t''}^t G_x(t,t')G_c(t',t'')\,\mathrm{d}t' \cdot G_c(t,t'')\right] \mathrm{d}t'' \\
\sigma_{vv} &= \frac{2D\gamma}{\tau^2}\int_0^t \left[\int_{t''}^t \frac{\partial}{\partial t}G_x(t,t')G_c(t',t'')\,\mathrm{d}t'\right]^2 \mathrm{d}t'' \\
\sigma_{vc} &= \frac{2D\gamma}{\tau^2}\int_0^t \left[\int_{t''}^t \frac{\partial}{\partial t}G_x(t,t')G_c(t',t'')\,\mathrm{d}t' \cdot G_c(t,t'')\right] \mathrm{d}t'' \\
\sigma_{cc} &= \frac{2D\gamma}{\tau^2}\int_0^t G_c^2(t,t')\,\mathrm{d}t'.
\end{aligned} \tag{17}$$

It can be easily shown that the probability distribution $W(x, v, c, t)$ is reduced to that for the GWN system in the limit of $\tau \to 0$. This is due to the fact that σ_{xc} and σ_{vc} vanish for $\tau \to 0$, $\sigma_{cc} \sim \frac{D\gamma}{\tau}$; as a result, the dynamics between x and c, and v and c become decoupled and integration with respect to c can be performed trivially, resulting in $W(x, v, t)$.

3. Dynamical behaviour of coordinate variance

From the formal solutions for $x(t)$, $v(t)$ and $c(t)$ expressed in terms of the Green function, the elements of the covariance matrix can be easily determined. However, a better approach is to utilize ordinary differential equations governing the motion of σ_{xx}. Below we derive the equation of motion. The following closed-system set of coupled differential equations for the elements of the covariance matrix can be obtained with the help of the well-known relation between the noise-averaged dynamical quantity and the Fokker–Planck equation in the adjoint form [18]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle f(x(t), v(t), c(t))\rangle = \left\langle \mathcal{L}_{\mathrm{FP}}^{\dagger} f(x(t), v(t), c(t))\right\rangle \tag{18}$$

where $\mathcal{L}_{\mathrm{FP}}^{\dagger}$ is the adjoint of the operator $\mathcal{L}_{\mathrm{FP}}$ expressed as

$$\mathcal{L}_{\mathrm{FP}}^{\dagger} = v\frac{\partial}{\partial x} - \alpha(t)x\frac{\partial}{\partial v} - \gamma v\frac{\partial}{\partial v} + c\frac{\partial}{\partial v} - \frac{1}{\tau}c\frac{\partial}{\partial c} + \frac{D\gamma}{\tau^2}\frac{\partial^2}{\partial c^2}. \tag{19}$$

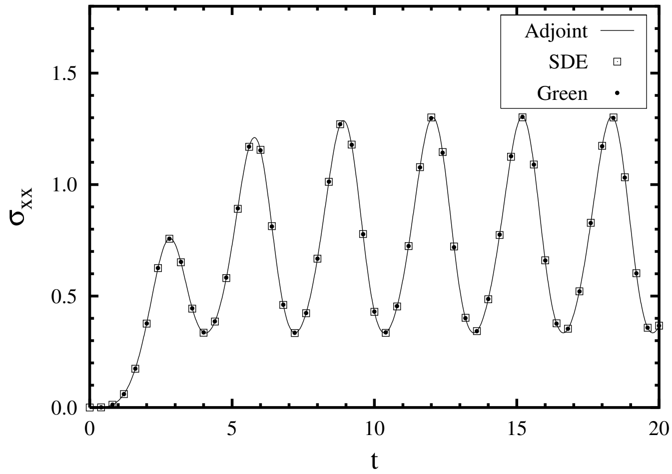

Figure 1. Variance σ_{xx} as a function of time at $\omega_0^2 = 1, D = 1, \tau = 1, \psi = 0$ and $\epsilon = 0.5$. The values obtained from equations (20) are depicted as a solid line, while circles and squares denote the results obtained from integration of the Green function (equation (17)) and numerical simulation of the stochastic differential equation (equation (21)), respectively.

Then the coupled differential equation for each element of the covariance matrix can be written as

$$\begin{aligned}
\dot{\sigma_{xx}} &= 2\sigma_{xv} \\
\dot{\sigma_{xv}} &= -\alpha(t)\sigma_{xx} - \gamma\sigma_{xv} + \sigma_{xc} + \sigma_{vv} \\
\dot{\sigma_{xc}} &= -\frac{1}{\tau}\sigma_{xc} + \sigma_{vc} \\
\dot{\sigma_{vv}} &= -2\alpha(t)\sigma_{xv} - 2\gamma\sigma_{vv} + 2\sigma_{vc} \\
\dot{\sigma_{vc}} &= -\alpha(t)\sigma_{xv} - \left(\gamma + \frac{1}{\tau}\right)\sigma_{vc} + \sigma_{cc} \\
\dot{\sigma_{cc}} &= -\frac{2}{\tau}\sigma_{cc} + \frac{2D\gamma}{\tau^2}.
\end{aligned} \tag{20}$$

Numerical solutions of equation (20) together with $\sigma_{xx}(t)$ obtained from numerical integration of equation (17) expressed in terms of the Green function are plotted in figure 1. The numerical integrations in both cases were carried out by a fourth-order Runge–Kutta algorithm with a time step of $\Delta t = 10^{-3}$. They show perfect agreement with each other. It is of interest to perform direct numerical simulations utilizing the stochastic differential equation. For the case of coloured noise whose dynamics is governed by equation (8), equation (1) can be rewritten in the following form:

$$\mathrm{d}\begin{bmatrix} x \\ v \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\alpha(t) & -\gamma & 1 \\ 0 & 0 & -1/\tau \end{bmatrix} \begin{bmatrix} x \\ v \\ c \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 0 \\ 0 \\ \sqrt{2D\gamma}/\tau \end{bmatrix} \xi_t \, \mathrm{d}t \tag{21}$$

where the ξ_t are standard Gaussian random variables for each t. The constant nature of the term $\sqrt{2D\gamma}/\tau$ enables us to use the explicit order 3.0 weak scheme [17] in a straightforward manner for the simulation of equation (21). σ_{xx} was evaluated over the numerically constructed 10^5 sample path. The simulation results are also depicted in figure 1. Figure 1 clearly shows that the bounded motion of σ_{xx} becomes asymptotically periodic in the long time limit irrespective

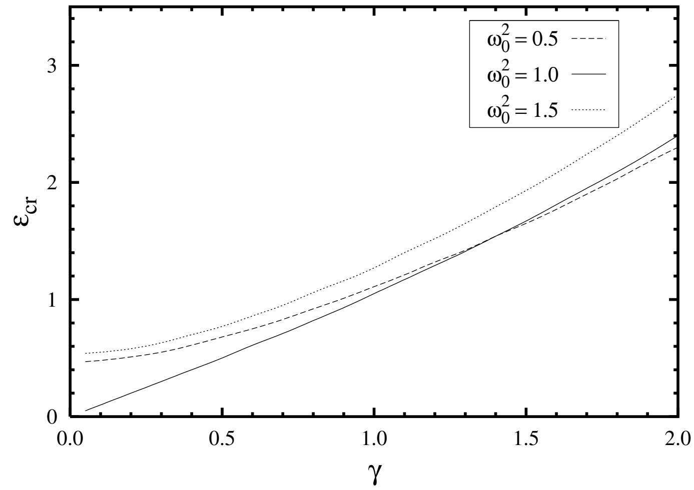

Figure 2. Critical strength of the modulation ϵ_{cr} as a function of the friction coefficient γ for $\tau = 1$ and $D = 1$.

of the initial conditions. On the other hand, the motion of σ_{xx} becomes unbounded for values of ϵ greater than ϵ_{cr}. Investigation of the dependence of σ_{xx} on the system parameters D, τ and γ reveals that ϵ does not depend on D and γ, whereas it grows as γ becomes larger (see figure 2). Equation (4) reveals that the point at which $x_h(t)$ becomes unbounded is identified by the relation

$$\mathrm{Im}\,\nu = \pm\frac{\gamma}{2}. \tag{22}$$

The physical parameters governing the Floquet parameter ν are ϵ, γ and ω_0^2. Thus, for fixed values of γ and ω_0^2, ν is a function of ϵ only and the critical value of ϵ which satisfies equation (22) can be easily obtained using Mathematica. It is found that this critical value of ϵ is identical to ϵ_{cr}, which confirms that when σ_{xx} no longer has an asymptotic stationary solution, the motion of $x_h(t)$ simultaneously becomes unbounded.

Next, the phase-averaged stationary long time values of σ_{xx}, $\langle\sigma_{xx}^{\mathrm{st}}\rangle_\psi$, are calculated numerically as a function of ϵ for different values of τ and ω_0^2 and the results are depicted in figures 3 and 4, respectively. Below $\langle\,\rangle_\psi$ indicates an average with respect to the phase assuming that the phase is uniformly distributed. The general trend of $\langle\sigma_{xx}^{\mathrm{st}}\rangle_\psi$ with varying ϵ is similar to that found for the GWN case (see figure 3), except that the suppression of the variance is enhanced as τ increases (see figure 4). However, as mentioned above, the value of ϵ_{cr} does not change as τ increases.

4. Analytical approximation in the small ϵ limit

As shown below, an analytical solution of $\langle\sigma_{xx}^{\mathrm{st}}\rangle_\psi$ can be obtained up to $\mathrm{O}(\epsilon^2)$ for small values of the modulation ϵ. From equation (20), the following coupled third-order differential equation for σ_{xx} and σ_{xc} can be obtained upon elimination of σ_{xv}, σ_{vv}, σ_{vc} and σ_{cc}:

$$\begin{aligned}&\dddot{\sigma}_{xx} + 3\gamma\ddot{\sigma}_{xx} + 2(\gamma^2 + 2\alpha(t))\dot{\sigma}_{xx} + 2(\alpha'(t) + 2\gamma\alpha(t))\sigma_{xx} - 6\dot{\sigma}_{xc} - 4\left(\gamma + \frac{1}{\tau}\right)\sigma_{xc} = 0\\ &\ddot{\sigma}_{xc} + \left(\gamma + \frac{2}{\tau}\right)\dot{\sigma}_{xc} + \left(\frac{1}{\tau^2} + \frac{\gamma}{\tau} + \alpha(t)\right)\sigma_{xc} = \frac{D\gamma}{\tau}.\end{aligned} \tag{23}$$

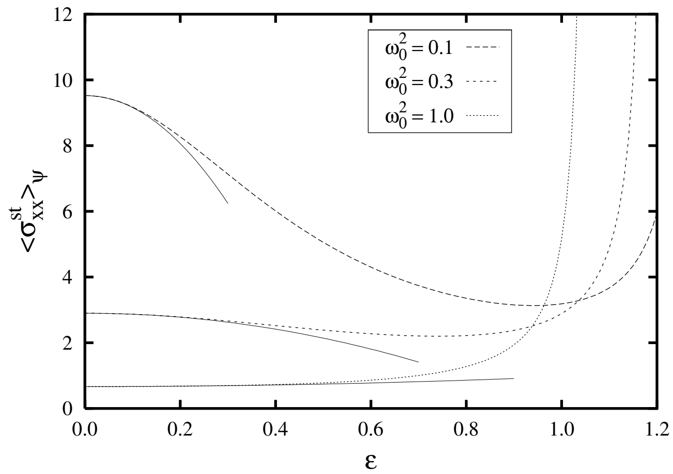

Figure 3. Phase-averaged displacement variance $\langle\sigma_{xx}^{\mathrm{st}}\rangle_\psi$ as a function of the modulation ϵ at $\gamma = 1, D = 1$ and $\tau = 1$ for different values of the angular frequency $\omega_0^2 = 0.1, 0.3, 1.0$. Numerically computed values are depicted by dotted lines, while solid lines show the analytical approximation for small ϵ.

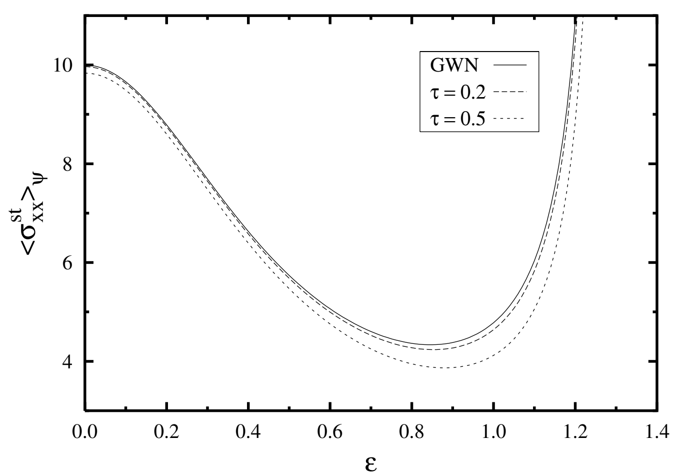

Figure 4. Phase-averaged displacement variance $\langle\sigma_{xx}^{\mathrm{st}}\rangle_\psi$ as a function of the modulation ϵ at $\gamma = 1, D = 1$ and $\omega_0^2 = 0.1$ for different values of the correlation length $\tau = 0.2, 0.5$. The solid line depicts the result of the GWN system.

Assuming that the solution is bounded, σ_{xx} is π-periodic in the asymptotic region $t \to \infty$. Then $\sigma_{xx}^{\mathrm{st}}(t)$ and $\sigma_{xc}^{\mathrm{st}}(t)$ can be expressed as a Fourier series

$$\sigma_{xx}^{\mathrm{st}}(t) = \sum_{n=-\infty}^{\infty} a_n \exp(\mathrm{i}n(2t+\psi)) \qquad \sigma_{xc}^{\mathrm{st}}(t) = \sum_{n=-\infty}^{\infty} b_n \exp(\mathrm{i}n(2t+\psi)). \tag{24}$$

The recurrence relations for a_n and b_n, can be obtained as follows:

$$\begin{aligned} &\epsilon A_n^- a_{n-1} + A_n a_n + \epsilon A_n^+ a_{n+1} + B_n b_n = 0 \\ &\epsilon b_{n-1} + C_n b_n + \epsilon b_{n+1} = \frac{D\gamma}{\tau}\delta_{n0} \end{aligned} \tag{25}$$

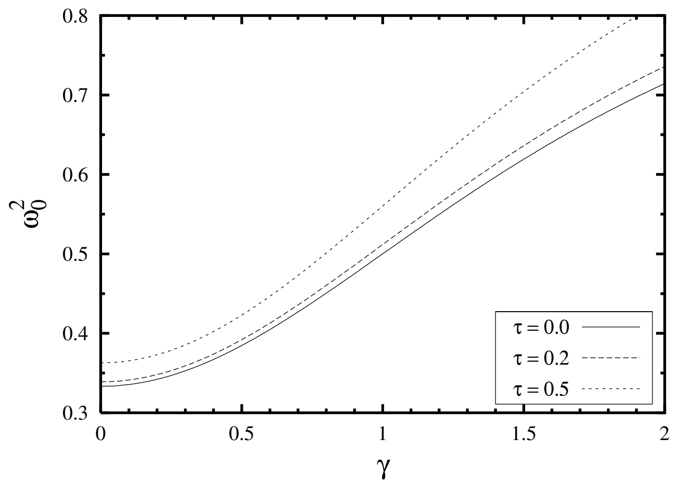

Figure 5. Plot of ω_0^2 as a function of γ satisfying $\Delta_{xx} = 0$ for various τ (see equation (28)). The area under each curve represents the region for which $\langle\sigma_{xx}^{\rm st}\rangle_\psi$ has a minimum.

where

$$
\begin{aligned}
A_n^- &= \gamma + (2n-1)\mathrm{i} \\
A_n &= \gamma\omega_0^2 - 3\gamma n^2 + \left(2\omega_0^2 + \gamma^2\right)n\mathrm{i} - 2n^3\mathrm{i} \\
A_n^+ &= \gamma + (2n+1)\mathrm{i} \\
B_n &= \left(\gamma + \frac{1}{\tau}\right) + 3n\mathrm{i} \\
C_n &= \omega_0^2 - 4n^2 + \frac{1}{\tau}\left(\gamma + \frac{1}{\tau}\right) + 2n\left(\gamma + \frac{2}{\tau}\right)\mathrm{i}.
\end{aligned}
\tag{26}
$$

Approximating the recurrence relation for $\sigma_{xx}^{\rm st}(t)$ and $\sigma_{xc}^{\rm st}(t)$ by the continued fraction in the small ϵ limit, we obtain the following solutions:

$$\left\langle\sigma_{xx}^{\rm st}\right\rangle_\psi = \frac{D}{\omega_0^2}\cdot\frac{1+\gamma\tau}{1+\gamma\tau+\omega_0^2\tau^2} + \Delta_{xx}\epsilon^2 + \mathrm{O}(\epsilon^4) \tag{27}$$

$$\Delta_{xx} = \frac{2D}{\omega_0^2}\cdot\frac{1+\gamma\tau}{1+\gamma\tau+\omega_0^2\tau^2}\,\mathrm{Re}\left(\frac{A_0^+A_1^-}{A_0A_1} + \frac{1}{C_0C_1} + \frac{A_0^+B_1}{A_1B_0}\right). \tag{28}$$

As the strength of the modulation goes to zero ($\epsilon \to 0$), $\left\langle\sigma_{xx}^{\rm st}\right\rangle_\psi$ approaches the coordinate variance of a harmonic oscillator driven by exponentially correlated coloured noise, $\frac{D}{\omega_0^2}\cdot\frac{1+\gamma\tau}{1+\gamma\tau+\omega_0^2\tau^2}$, as the correlation length of the noise τ goes to zero. This coordinate variance, in turn, is reduced to that of a harmonic oscillator driven by GWN, $\frac{D}{\omega_0^2}$.

The condition for $\left\langle\sigma_{xx}^{\rm st}\right\rangle_\psi$ to have a minimum is $\Delta_{xx} < 0$. To find the region of ω_0 which satisfies the condition that $\left\langle\sigma_{xx}^{\rm st}\right\rangle_\phi$ has a minimum, we depict the region of ω_0^2 satisfying $\Delta_{xx} < 0$ as a function of γ for various values of τ. Figure 5 clearly shows that τ not only enhances the suppression of the coordinate variance but also expands the region which satisfies the condition for $\left\langle\sigma_{xx}^{\rm st}\right\rangle_\psi$ to have a minimum.

5. Stability analysis based on the Floquet spectrum

Finally, we consider the Floquet eigenvalue μ defined from the Fokker–Planck operator. A differential operator that varies periodically with time implies a Floquet-type solution in which the Floquet exponent governs the stability of the system. Since the Fokker–Planck operator $\mathcal{L}_{\mathrm{FP}}$ is π-periodic, the probability density function $W(x, v, c, t)$ can be expressed as a linear combination of normal solutions.

$$W = \sum_{\mu} c_{\mu}\,\mathrm{e}^{-\mu t}\, p_{\mu}(x, v, c, t) \tag{29}$$

where p_{μ} is a π-periodic function and μ is the Floquet eigenvalue satisfying the following eigenvalue equation:

$$\left(\mathcal{L}_{\mathrm{FP}} - \frac{\partial}{\partial t}\right) p_{\mu} = -\mu p_{\mu}. \tag{30}$$

Transforming to an adjoint form, we obtain

$$\left(\mathcal{L}^{\dagger}_{\mathrm{FP}} + \frac{\partial}{\partial t}\right) p^{\dagger}_{\mu} = -\mu p^{\dagger}_{\mu} \tag{31}$$

where $p^{\dagger}_{\mu}$ can be expressed as a polynomial with respect to x, v and c. The trivial solution is $p^{\dagger} = 1$ for the eigenvalue $\mu = 0$. The first nonzero eigenvalue can be obtained from $p^{\dagger} = a_1(t)x + a_2(t)v + a_3(t)c$. Since the Fokker–Planck equation is invariant under parity transformation, $p^{\dagger}$ does not contain a constant term. $p^{\dagger}$ satisfies equation (31), therefore

$$\dot{a}_1 + \mu a_1(t) - \alpha(t) a_2(t) = 0 \tag{32}$$

$$\dot{a}_2 + (\mu - \gamma) a_2 + a_1 = 0 \tag{33}$$

$$\dot{a}_3 + \left(\mu - \frac{1}{\tau}\right) a_3 + a_2 = 0. \tag{34}$$

Substituting equation (33) into equation (32), we obtain the following differential equation for a_2:

$$\ddot{a}_2 + (2\mu - \gamma)\dot{a}_2 + [\mu(\mu - \gamma) + \alpha(t)] a_2 = 0 \tag{35}$$

and a_2 and a_3 can be written as

$$a_2(t) = c_1 \exp(\mathrm{i}\nu(t + \psi/2)) \exp(-(\mu - \gamma/2)t)\, p\left(t + \frac{\psi}{2}\right) \\ + c_2 \exp(-\mathrm{i}\nu(t + \psi/2)) \exp(-(\mu - \gamma/2)t)\, p\left(-t - \frac{\psi}{2}\right) \tag{36}$$

$$a_3(t) = c_3 \exp(-(\mu - 1/\tau)t) - \int_0^t \exp(-(\mu - 1/\tau)(t - s)) a_2(s)\, \mathrm{d}s \tag{37}$$

where ν is the Floquet parameter in equation (4). Since $a_2(t)$ and $a_3(t)$ are π-periodic, the following three cases are possible for μ:

$$\mu_{100} = -\mathrm{i}\nu + \frac{\gamma}{2} \qquad \mu_{010} = \mathrm{i}\nu + \frac{\gamma}{2} \qquad \mu_{001} = \frac{1}{\tau}. \tag{38}$$

It is then straightforward to show that the Floquet spectrum μ_{nml} for each value of n, m and l is

$$\mu_{nml} = n\mu_{100} + m\mu_{010} + l\mu_{001} \qquad (n, m, l = 0, 1, 2, \cdots) \tag{39}$$

where μ_{100}, μ_{010} and μ_{001} are defined in equation (38). Equation (38) defines the relation between the Floquet eigenvalues of the Fokker–Planck operator and the Floquet parameter that determines the stability of the damped Mathieu equation. μ_{100} and μ_{010} govern the stability of the system and they remain the same as those for the GWN system. The additional exponent $1/\tau$ controls the decreasing rate of the amplitude of the noise variance σ_{cc}. Hence the correlation length of the noise does not change the condition for the stability of the solution.

6. Conclusions

We have developed a general scheme to study the stochastic dynamics of a linear parametric oscillator driven by coloured noise. It was our intention throughout the analytical and numerical investigation to understand the change in dynamical behaviour as the external noise changes from white to coloured. Specifically, we found that the suppression of the coordinate variance $\langle\sigma_{xx}^{\mathrm{st}}\rangle_\psi$ increases with increasing correlation length of the noise τ. On the other hand, the critical strength of parametric modulation at which the motion becomes unbounded is independent of τ.

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