Using Bayesian spectral reprojection to resolve the Gibbs phenomenon

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Introduction

Fourier samples are collected in a variety of applications including magnetic resonance imaging (MRI) and synthetic aperture radar (SAR). In an idealized situation, recovering images from partial Fourier data may be done simply and efficiently by using the inverse fast Fourier transform (FFT). However, the quality of the solution deteriorates in practice due to the facts that

- the data acquisition system is **noisy**;
- the underlying function is **not periodic**.

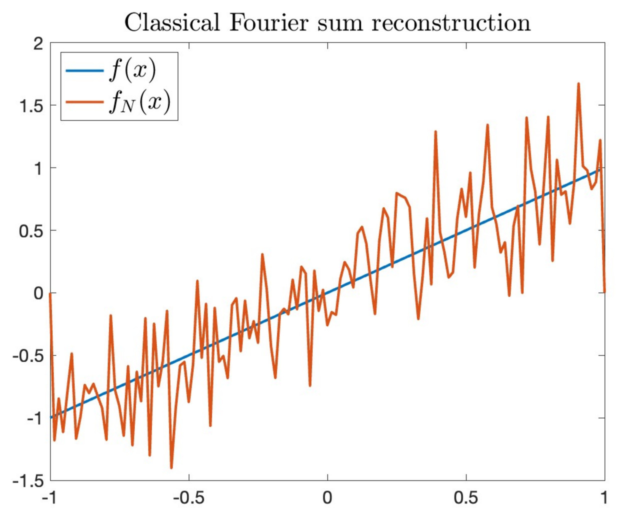

Figure 1: The function $f(x) = x$ and its Fourier partial sum $f_N(x)$ with $N = 128$.

Here we use a **Bayesian approach** to construct a posterior using the given **Fourier data** and a prior based on properties of **orthogonal polynomials**.

Problem set up

Consider a real-valued L^2 integrable function f on $[-1, 1]$:

$$f : [-1, 1] \to \mathbb{R} \quad \text{s.t.} \quad \int_{-1}^{1} |f(x)|^2 dx < \infty.$$

We are given noisy Fourier measurements $\widehat{\mathbf{b}} = \left(\widehat{b}_k : -\frac{N}{2} \leq k < \frac{N}{2}\right) \in \mathbb{C}^N$ such that

$$\widehat{b}_k = \frac{1}{2}\int_{-1}^{1} f(x) e^{-ik\pi x} dx + \epsilon_k$$

where $\boldsymbol{\epsilon} = \left(\epsilon_k : -\frac{N}{2} \leq k < \frac{N}{2}\right) \in \mathbb{C}^N$ corresponds to a typically unknown noise vector.
Our **goal** is to construct point values of the function f.

Model problem

We construct the *linear inverse problem*

$$\widehat{\mathbf{b}} = \mathbf{F}\,\mathbf{f} + \boldsymbol{\epsilon} \tag{1}$$

to recover function f on a set of uniform points. In particular,

- $\widehat{\mathbf{b}} \in \mathbb{C}^N$ is the given Fourier measurements vector;
- $\mathbf{F} \in \mathbb{C}^{N \times N}$ is the discrete Fourier transform matrix;
- $\mathbf{f} = \big(f(x_j) : 0 \leq j < N\big) \in \mathbb{R}^N$ is the vector of unknowns corresponding to the function evaluation on the uniform grid points $x_j = -1 + \frac{2j}{N}$, $j = 0, 1, ..., N-1$;
- $\boldsymbol{\epsilon} \in \mathbb{C}^N$ is the complex noise vector.

Numerical approximation by orthogonal polynomials

The function $f(x)$ can by approximated by its the Gegenbauer expansion based on the first $m+1$ terms

$$f_m^\lambda(x) = \sum_{l=0}^{m} \widehat{f}_l^\lambda C_l^\lambda(x), \tag{2}$$

where its Gegenbauer coefficients $\widehat{f}_l^\lambda$ are given by

$$\widehat{f}_l^\lambda = \frac{1}{h_l^\lambda}\int_{-1}^{1} (1-x^2)^{\lambda - \frac{1}{2}} f(x)\, C_l^\lambda(x) dx. \tag{3}$$

The hierarchical Bayesian model

posterior $\propto$ likelihood $\times$ prior $\times$ hyper-prior

Likelihood Fourier data

$$\begin{pmatrix} \mathrm{Re}\,(\widehat{\mathbf{b}}) \\ \mathrm{Im}\,(\widehat{\mathbf{b}}) \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mathrm{Re}\,(\mathbf{F}) \\ \mathrm{Im}\,(\mathbf{F}) \end{pmatrix} \mathbf{f}, \alpha^{-1}\mathbf{I}_{2N}\right),$$

$$\widetilde{\mathbf{b}} \sim \mathcal{N}(\widetilde{\mathbf{F}}\,\mathbf{f}, \alpha^{-1}\mathbf{I}_{2N}). \tag{4}$$

$$p(\widetilde{\mathbf{b}}\,|\,\mathbf{f}, \alpha) = (2\pi)^{-2N/2}\alpha^{2N/2}\exp\left\{-\frac{\alpha}{2}\|\widetilde{\mathbf{F}}\,\mathbf{f} - \widetilde{\mathbf{b}}\|_2^2\right\}. \tag{5}$$

Prior Numerical approximation

$$\mathbf{f} \sim \mathcal{N}(\mathbf{f}_m^\lambda, \beta^{-1}\mathbf{I}_N),$$

$$\mathbf{f} \sim \mathcal{N}(\mathbf{G}\,\mathbf{f}, \beta^{-1}\mathbf{I}_N). \tag{6}$$

$$p(\mathbf{f}\,|\,\beta) = (2\pi)^{-N/2}\beta^{N/2}\exp\left\{-\frac{\beta}{2}\|(\mathbf{I}_N - \mathbf{G})\,\mathbf{f}\|_2^2\right\}. \tag{7}$$

Hyper-prior

$$p(\alpha) = \Gamma(\alpha\,|\,c, d) = \frac{d^c}{\Gamma(c)}\alpha^{c-1}\exp(-d\,\alpha), \tag{8}$$

$$p(\beta) = \Gamma(\beta\,|\,c, d) = \frac{d^c}{\Gamma(c)}\beta^{c-1}\exp(-d\,\beta). \tag{9}$$

Posterior from Bayesian inference

$$p(\mathbf{f}, \alpha, \beta\,|\,\widetilde{\mathbf{b}}) \propto p(\widetilde{\mathbf{b}}\,|\,\mathbf{f}, \alpha)\,p(\mathbf{f}\,|\,\beta)\,p(\alpha)\,p(\beta). \tag{10}$$

$$p(\mathbf{f}, \alpha, \beta\,|\,\widetilde{\mathbf{b}}) \propto \alpha^{c+N-1}\beta^{c+N/2-1}\exp\left\{-\frac{\alpha}{2}\|\widetilde{\mathbf{F}}\,\mathbf{f} - \widetilde{\mathbf{b}}\|_2^2 - \frac{\beta}{2}\|\mathbf{I}_n(\mathbf{f}_{m,N}^\lambda - \mathbf{f})\|_2^2 - d\,\alpha - d\,\beta\right\}. \tag{11}$$

Numerical scheme: Bayesian coordinate descent (BCD)

The **maximum a posterior (MAP) estimate** could be obtained equivalently as minimizing the negative logarithm of the posterior, that is, the objective function is

$$\mathcal{G}(\mathbf{f}, \alpha, \beta) = -(c+N-1)\log(\alpha) - (c + \frac{N}{2} - 1)\log(\beta) + \frac{\alpha}{2}\|\widetilde{\mathbf{F}}\,\mathbf{f} - \widetilde{\mathbf{b}}\|_2^2 + \frac{\beta}{2}\|(\mathbf{I}_N - \mathbf{G})\mathbf{f}\|_2^2 + d\,\alpha + d\,\beta.$$

Algorithm 1 BCD algorithm for the MAP estimate

Initialize $\alpha^{(0)}, \beta^{(0)}$
for $l = 1$ to Max$_{\text{iter}}$ **do**
  Update $\mathbf{f}$ by solving $(\alpha^{(l)}\,\widetilde{\mathbf{F}}^T\widetilde{\mathbf{F}} + \beta^{(l)}\,(\mathbf{I}_N - \mathbf{G})^T(\mathbf{I}_N - \mathbf{G}))\mathbf{f}^{(l+1)} = \alpha^{(l)}\,\widetilde{\mathbf{F}}^T\,\widetilde{\mathbf{b}}$
  Update α by setting $\alpha^{(l+1)} = \dfrac{2(c+N-1)}{\|\widetilde{\mathbf{F}}\,\mathbf{f}^{(l+1)} - \widetilde{\mathbf{b}}\|_2^2 + 2d}$
  Update β by setting $\beta^{(l+1)} = \dfrac{2c+N-2}{\|(\mathbf{I}_N - \mathbf{G})\mathbf{f}^{(l+1)}\|_2^2 + 2d}$
  Break if convergence criterion is satisfied
end for

Note: The algorithm above provides one approach to solve for the MAP estimate. Indeed the whole posterior distribution could be characterized based on the results of Bayesian inference.

Numerical results

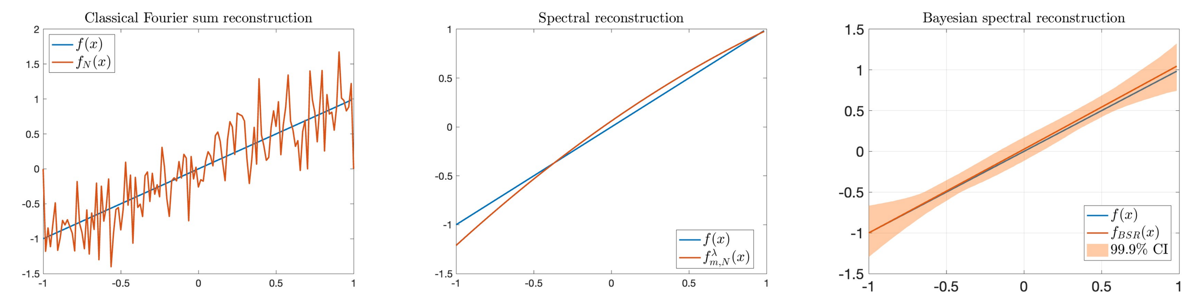

Figure 2: The function $f(x) = x$ and its reconstructions.

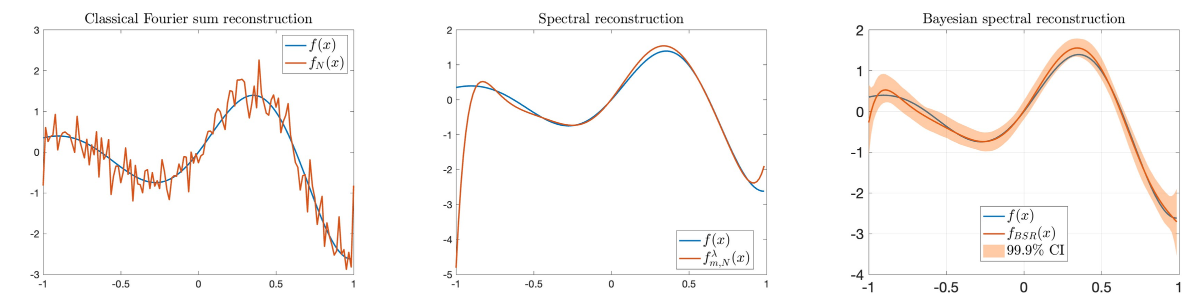

Figure 3: The function $f(x) = e^x \sin(5x)$ and its reconstructions.

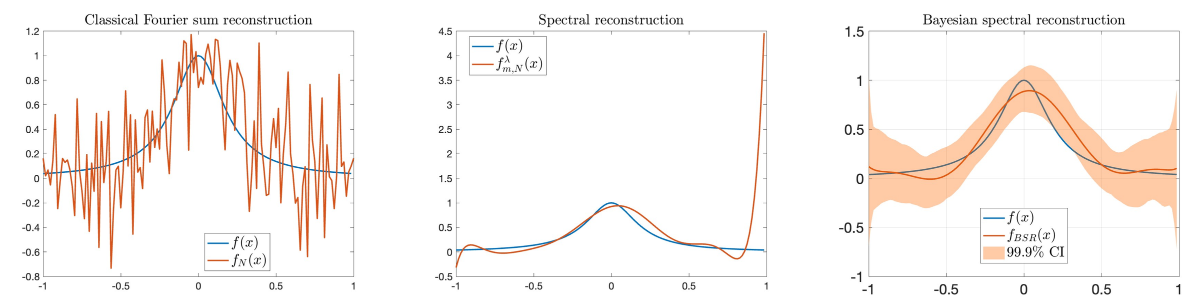

Figure 4: The function $f(x) = \frac{1}{1+25x^2}$ and its reconstructions.

- More accurate reconstruction;
- Resolution of Gibbs phenomenon and Runge phenomenon;
- Uncertainty quantification.

Future directions

- Theoretical analysis on the connection with spectral reprojection;
- Extension to the case of incomplete data;
- Inference on the auto selection of numerical parameters;
- Application on 2D images.

References

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