# Lippman-Schwinger-Lanczos algorithm for inverse scattering problems.

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- Reduced Order Models (ROMs) for forward problems: If e.g. PDE is linear, find a low dimensional matrix that acts like the differential operator.
- ROMs for inverse problems: Given data, find a ROM which matches the data, use this ROM to extract the unknown coefficient.

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- Time domain reconstruction of wave speed (Borcea, Garnier, Mamonov, Zimmerling 2022 )

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- Druskin, V. , Mamonov, A. and Zaslavsky, M., A nonlinear method for imaging with acoustic waves via reduced order model backprojection, SIAM Journal on Imaging Sciences, (2018).
- Borcea, L., Druskin, V., and Mamonov, A., Zaslavsky, M. and Zimmerling, J., Reduced Order Model Approach to Inverse Scattering, SIAM Journal on Imaging Sciences, (2020).

# Time domain SISO problem

$$u_{tt} + Au = 0 \text{ in } \Omega \times [0, \infty) \quad (1)$$
$$u(t = 0) = g \text{ in } \Omega \quad (2)$$
$$u_t(t = 0) = 0 \text{ in } \Omega \quad (3)$$

where

$$A = A_0 + q \quad (4)$$

- $A_0 \geq 0$ is known background, (for example $A_0 = -\Delta$),
- $q(x) \geq 0$ is our unknown potential
- initial data $g$ is localized (approximate delta) source
- assume homogeneous Neumann boundary conditions on the spatial boundary $\partial\Omega$.

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- The inverse problem is as follows: Given

$$\{F(k\tau)\} \quad \text{for} \quad k = 0, \ldots, 2n-2,$$

reconstruct $q$.

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$$M_{kl} = \int_\Omega u_k u_l dx \tag{7}$$

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- from the cosine angle sum formula

$$M_{kl} = \frac{1}{2} \left( F((k-l)\tau) + F((k+l)\tau) \right), \tag{9}$$

$M$ can be obtained directly from the data.

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- . The functions $\{v_k\}$ will be orthonormal in the $L^2$ norm (Gram-Schmidt).

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- This is because we start with a local source, orthogonalize sequentially, reflections overlap with previous times.
- So do all of the above for the known background problem

# Time domain SISO problem

- Background exact solution

$$u^0(x,t) = \cos(\sqrt{A_0}t)g(x). \tag{11}$$

and snapshots $\{u_j^0\}$

- mass matrix

$$M_{kl}^0 = \int_\Omega u_k^0 u_l^0 dx, \tag{12}$$

- Cholesky decomposition

$$M^0 = (U^0)^\top U^0,$$

- orthogonalized background snapshots

$$\vec{v}^0 = \vec{u}^0 (U^0)^{-1}. \tag{13}$$

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- Definition of our data generated snapshots

$$\begin{aligned} \vec{\mathbf{u}} &:= \vec{v}^0 U \\ &= \vec{u}^0 (U^0)^{-1} U. \end{aligned} \tag{15}$$

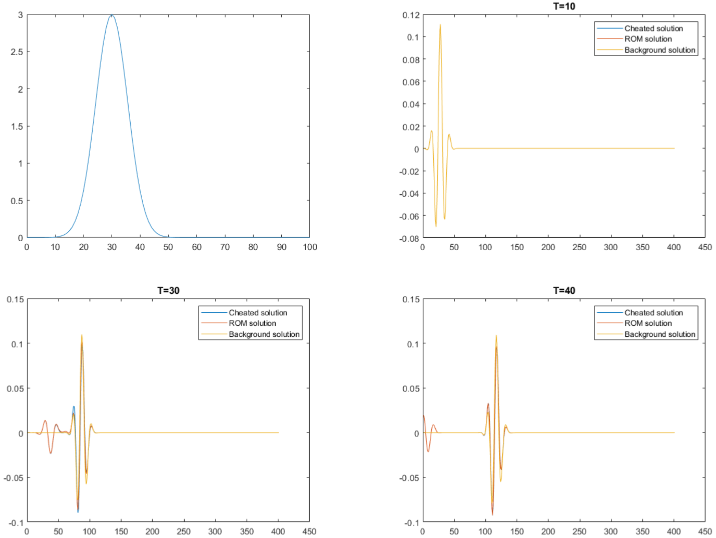


Figure: Data generated internal snapshots

- Time domain Lippmann-Schwinger

$$F_0(k\tau) - F(k\tau) = \int_0^{k\tau} \int_\Omega u_0(x, k\tau - t) u(x,t) q(x) dx dt. \qquad (16)$$

# Lippmann-Schwinger-Lanczos equation

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- Use data generated internal solution (interpolated in time)

$$F_0(k\tau) - F(k\tau) = \int_0^{k\tau} \int_\Omega u_0(x, k\tau - t) \mathbf{u}(x,t) q(x) dx dt \qquad (17)$$

# Spectral domain SISO problem

- Given $u$ such that

$$\begin{aligned} -u'' + q(x)u + \lambda u &= 0 \quad \text{for} \quad x \quad \text{on} \quad (0,1) \\ -u'(0) &= 1 \\ u(1) &= 0 \end{aligned}$$

- Define the transfer function $F(\lambda) := u(0;\lambda)$.
- Consider the inverse problem: Given $\{F(\lambda), F'(\lambda) : \lambda = b_1, \dots b_m\}$ , find $q(x)$

# Spectral domain SISO problem.

- Consider the inverse problem: Given $\{F(\lambda), F'(\lambda) : \lambda = b_1, \dots b_m\}$ , find $q(x)$
- Given $2m$ spectral data values to reconstruct $q(x)$
- Can do a modified version of what follows for other forms of spectral data
- We will construct a ROM that matches this data exactly

# Spectral domain SISO

- Consider exact solutions to above $u_1, \ldots, u_m$ corresponding to spectral points $\lambda = b_1, \ldots b_m$. and the subspace

$$G = \text{span}\{u_1, \ldots, u_m\}$$

- Although we do not know these solutions, we can obtain the Galerkin system (ROM) from the data
- Given by the mass and stiffness matrices

$$M_{ij} = \int_0^1 u_i u_j$$

and

$$S_{ij} = \int_0^1 u_i' u_j' + \int_0^1 q u_i u_j.$$

They are given by the formulas

$$M_{ij} = \frac{F(\lambda_i) - F(\lambda_j)}{\lambda_j - \lambda_i}, \quad M_{ii} = -\frac{dF}{d\lambda}(\lambda_i). \tag{18}$$

and

$$S_{ij} = \frac{F(\lambda_j)\lambda_j - F(\lambda_i)\lambda_i}{\lambda_j - \lambda_i}, \quad S_{ii} = \frac{d(\lambda F)}{d\lambda}(\lambda_i). \tag{19}$$

# Spectral domain SISO

- Searching for the unknown coefficients $\{c_i\}$ for the solution

$$u_G = \Sigma_{i=1}^m c_i u_i$$

$$M_{ij} = \int_0^1 u_i u_j$$

and

$$S_{ij} = \int_0^1 u_i' u_j' + \int_0^1 q u_i u_j.$$

- For forward solution would solve $(S + \lambda M)\vec{c} = \vec{F}$ where $F_i = F(b_i) = u_i(0)$.

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- Why does this work in the spectral domain?
- because the Krylov subspaces *are the same as* those generated by ROM- projected sequential time snapshots.

# Spectral domain SISO

- That is, if $d \in \mathbb{R}^m$ satisfies the Galerkin problem

$$Sd(t) + Md(t)_{tt} = 0, \qquad d(0) = b, \qquad d_{t\,t=0} = 0,$$

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- Then $d(\tau i)$ satisfy the second order finite-difference scheme

$$d[\tau(i+1)] = (2I - \tau A)d[\tau i] - d[\tau(i-1)], i = i, \ldots, m-1,$$

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- span$\{d(\tau i)\}$ are the *same* as the above Krylov subspaces w/ powers of $A$.

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- So the entries of this orthogonalized reduced order model (which can be obtained from the data) are the entries of the stiffness matrix

$$\hat{S}_{ij} = \int \hat{u}_i' \hat{u}_j' + \int_0^1 q \hat{u}_i \hat{u}_j$$

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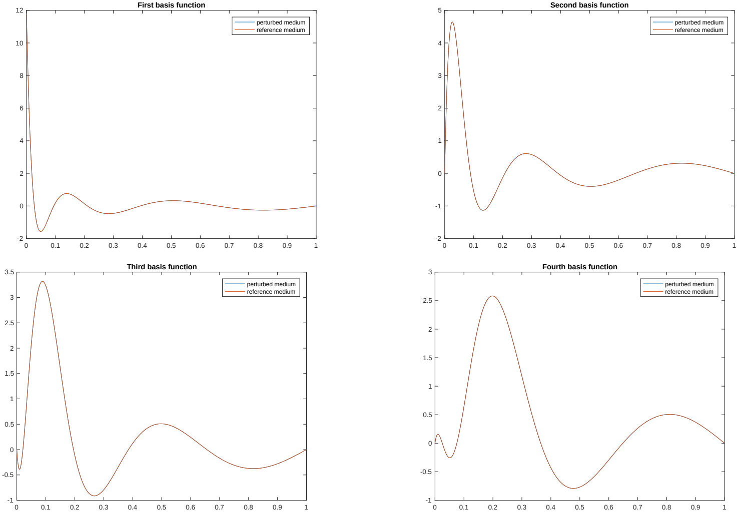
- correspond to sequentially orthogonalized projected time snapshots, which *depend only very weakly on the coefficient* .

# Weak dependence of orthogonalized spectral basis on $q$ (positive $\lambda = b_i$)



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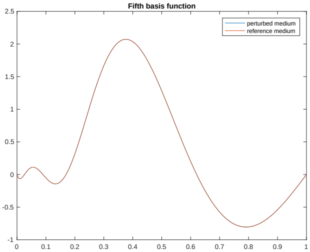


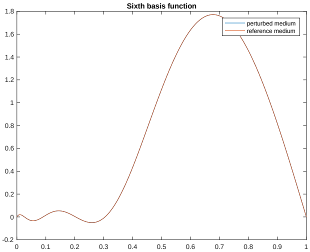

# Spectral domain Lippmann-Schwinger Lanczos approach

- Since the basis depends weakly on the unknown coefficient, we can get a data generated internal solution

$$\mathbf{u} = \sqrt{b^* M^{-1} b} V_0 Q_0 (A + \lambda I)^{-1} e_1$$

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- With data generated ROM we replace by $\mathbf{u}$ , the data generated internal solution.

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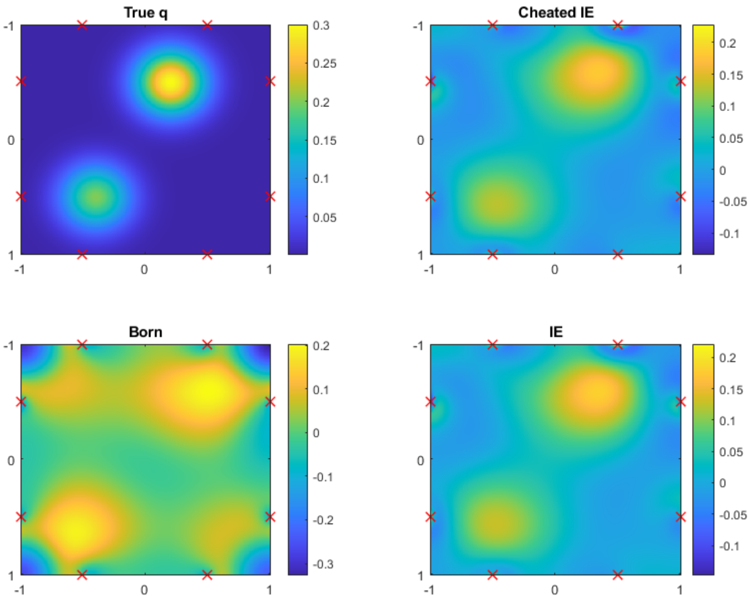

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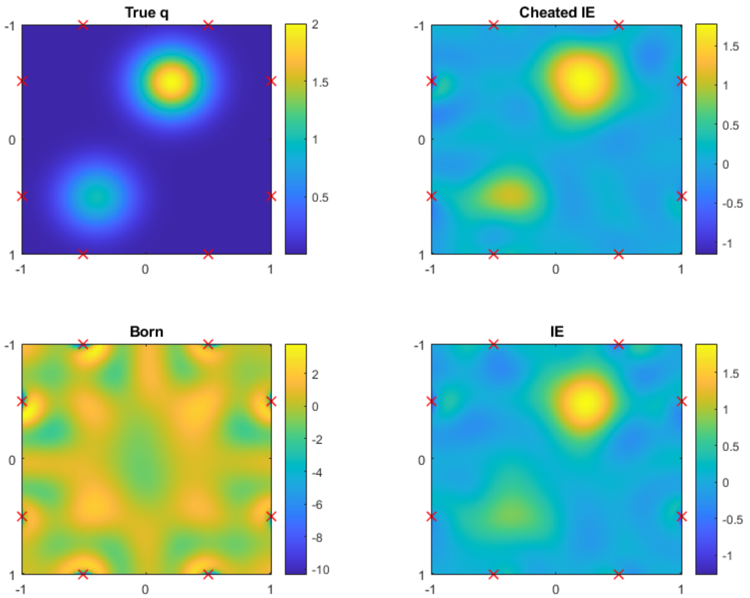

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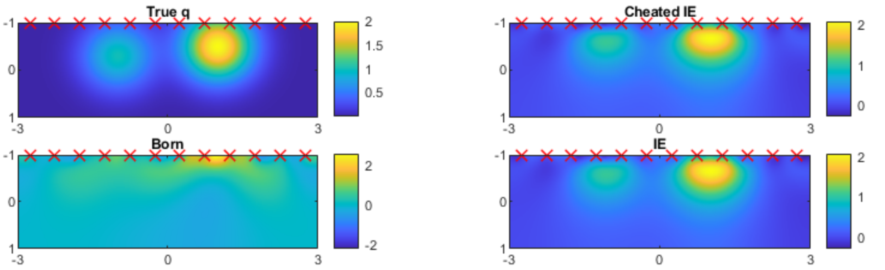



Figure: Experiment 3: True medium (top left) and its reconstructions using 'Cheated IE' (top right), Born linearization (bottom left) and our approach (bottom right)

# symmetric data: Lippman-Schwinger Lanczos approach

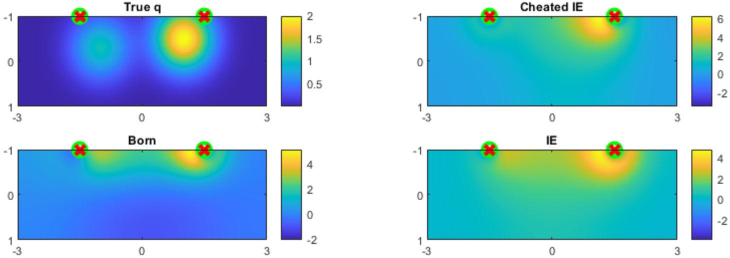
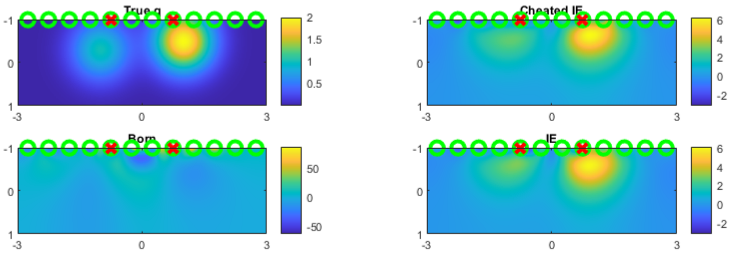



Figure: Experiment 1: True medium (top left) and its reconstructions using 'Cheated IE' (top right), Born linearization (bottom left) and our approach (bottom right)

# Non-symmetric data: Lippman-Schwinger Lanczos approach

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# Helmholtz 2d (with E. Cherkaev and J. Baker)

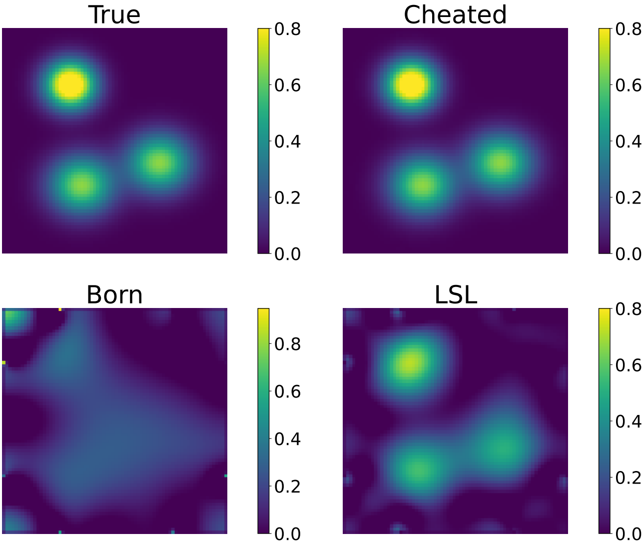


Figure: 2-D Helmholtz (positive $\lambda$)

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- In both cases the new basis is close to that from reference medium.
- Can use the reference medium basis to obtain approximations of internal solutions from data only
- Lippmann-Schwinger-Lanczos: use these internal solutions in Lippmann Schwinger, extendable to more general data sets.