Tropicalization of graph profiles for some classes of trees

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Abstract

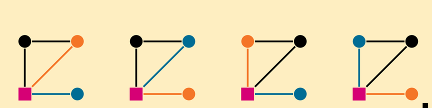
Many important problems in extremal combinatorics can be stated as inequalities of graph homomorphism numbers. For a fixed collection of graphs $\mathcal{U}$, the *tropicalization of the graph profile of* $\mathcal{U}$ essentially records all valid binomial inequalities involving graph homomorphism numbers for graphs in $\mathcal{U}$.

Building upon ideas and techniques described by Blekherman and Raymond in 2022, I present progress toward finding the tropicalization for some classes of trees.

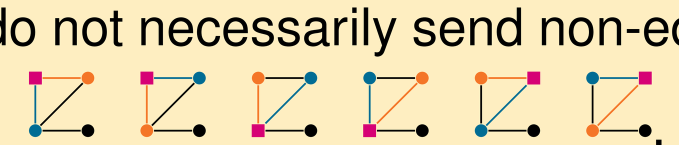
Graph Homomorphisms

Let F and G be graphs. A *graph homomorphism from F to G* is a map from $V(F)$ to $V(G)$ where we send edges of F to edges of G.

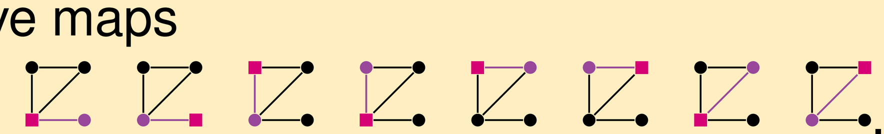
Example: Consider the homomorphisms from to .
This includes injective maps that additionally send non-edges to non-edges,



injective maps that do not necessarily send non-edges to non-edges,



and non-injective maps



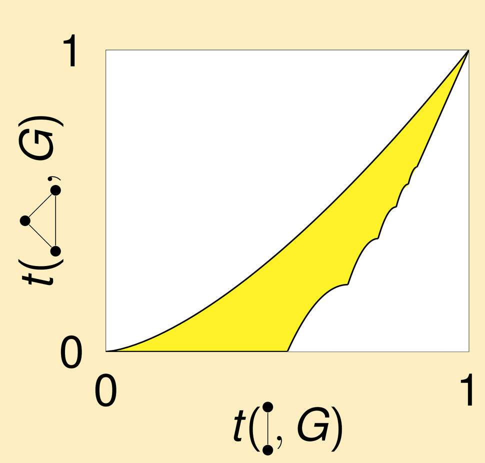
Let hom(F, G) denote the number of graph homomorphisms from F to G. Thus hom$(,) = 18$. Note that the total number of maps from $V(F)$ to $V(G)$ is $|V(G)|^{|V(F)|}$, so there are $4^3 = 64$ maps in this case.

Graph Profile

The (number) graph profile $\mathcal{N}_\mathcal{U}$ of $\mathcal{U}$ is

$$\mathcal{N}_\mathcal{U} := \{(\hom(F_1, G), \dots, \hom(F_\ell, G)) | G \text{ is a graph}\}.$$

$\mathcal{N}_\mathcal{U}$ encodes all valid polynomial inequalities in graph homomorphism numbers for graphs in the collection $\mathcal{U}$.



Density graph profile for $\mathcal{U} = \{ , \}$ (Razborov 2008)

Pictured is the *density* graph profile of the edge and the triangle $\left(t(F;G) = \frac{\hom(F,G)}{|V(G)|^{|V(F)|}} = \frac{\hom(F,G)}{\hom(\bullet,G)^{|V(F)|}}\right)$. Number profiles are more general than density profiles when $\mathcal{U}$ contains the vertex. We know very few profiles for pairs of graphs and no profiles for non-trivial triples.

Tropicalization

Using tropicalization, a tool from algebraic geometry, we can simplify the graph profile significantly, and it turns out that by doing so we can consider only the valid pure binomial inequalities.

Let $\log : \mathbb{R}^\ell_{>0} \to \mathbb{R}^\ell$ be defined as $\log(\mathbf{v}) := (\log(v_1), \dots, \log(v_l))$. For a set $\mathcal{S} \subset \mathbb{R}^\ell_{\geq 0}$, we define $\log(\mathcal{S}) := \log(\mathcal{S} \cap \mathbb{R}^\ell_{>0})$.

The tropicalization of $\mathcal{S}$ is defined to be:

$$\operatorname{trop}(\mathcal{S}) := \lim_{\tau\to\infty} \log_\tau(\mathcal{S}).$$

Theorem (Blekherman, Raymond, Singh, Thomas 2022)

The tropicalization trop$(\mathcal{N}_\mathcal{U})$ = cl(cone(log$(\mathcal{N}_\mathcal{U})$)) is a closed convex cone in $\mathbb{R}^\ell_{\geq 0}$ determined by linear inequalities corresponding to the pure binomial inequalities valid on $\mathcal{N}_\mathcal{U}$.

For example, the pure binomial inequality

$$\hom(, G)^3 - \hom(, G)^2 \geq 0$$

which is valid on $\mathcal{N}_{\{ , \}}$ becomes the valid linear inequality

$$3\log(\hom(, G)) - 2\log(\hom(, G)) \geq 0$$

for trop$(\mathcal{N}_{\{ , \}})$.

Previous Results

Theorem (Kopparty, Rossman 2011)

A linear program can compute

$$\max\{c \in \mathbb{R} \mid \hom(F_1, G) \geq \hom(F_2, G)^c \text{ for all graphs } G\}$$

when F_1 and F_2 are chordal and series-parallel graphs. In particular, this applies when F_1 and F_2 are trees.

Theorem (Blekherman, Raymond 2022)

trop$(\mathcal{N}_\mathcal{U})$ is a rational polyhedral cone when $\mathcal{U}$ contains graphs that are chordal and series-parallel.

Due to Blekherman and Raymond 2022, we have the explicit tropicalizations for the graph profiles of even cycles, odd cycles, complete graphs, paths, and stars; the latter is shown below.

Theorem (Blekherman, Raymond 2022)

Let $\mathcal{U} = \{S_0, S_1, \dots, S_m\}$ where S_i is the star graph with i branches. Then

$$\operatorname{trop}(\mathcal{N}_\mathcal{U}) = \left\{ y \in \mathbb{R}^{m+1} \middle| \begin{array}{l} -y_1 + y_2 \geq 0 \\ y_0 + y_{m-1} - y_m \geq 0 \\ y_{i-1} - 2y_i + y_{i+1} \geq 0 \quad \forall 1 \leq i \leq m-1 \\ m \cdot y_{m-1} - (m-1)\cdot y_m \geq 0 \end{array} \right\}.$$

The set trop$(\mathcal{N}_\mathcal{U})$ is the double hull of the following doubly extreme rays: $\vec{1}$, $(1, 0, 0, \dots, 0)$, $(1, 1, 2, 3, \dots, m)$, and $(1, 2, 3, 4, \dots, m+1)$.

Our Results (Dascălu, Raymond 2023+)

Let $\mathcal{U} = \{\bullet, , , , , \dots, S_{2,1^{m-1}}\}$ where $S_{2,1^k}$ is the star-like graph with one branch containing two edges and k branches containing one edge. Then

$$\operatorname{trop}(\mathcal{N}_\mathcal{U}) = \left\{ y \in \mathbb{R}^{m+1} \middle| \begin{array}{l} -y_1 + y_2 \geq 0 \\ 4y_1 - 3y_2 \geq 0 \\ 3y_1 - 3y_3 + y_4 \geq 0 \\ y_1 + 2y_{m-1} - 2y_m \geq 0 \\ y_0 + y_{m-1} - y_m \geq 0 \\ y_0 - 2y_1 + y_3 \geq 0 \\ y_{i-1} - 2y_i + y_{i+1} \geq 0 \quad \forall 2 \leq i \leq m-1 \\ m \cdot y_{m-1} - (m-1)\cdot y_m \geq 0 \end{array} \right\}.$$

The set trop$(\mathcal{N}_\mathcal{U})$ is the double hull of the following five doubly extreme rays: $\vec{1}$, $(1, 0, 0, \dots, 0)$, $(2, 2, 3, 4, \dots, m)$, $(3, 4, 5, \dots, m+2)$, $(4, 5, 7, \dots, 2m+1)$.

Further Questions

- Find the tropicalizations of the graph profiles of trees containing paths up to length 3, of other star-like graphs, and of other interesting classes of trees.
- Conjecture (Blekherman, Raymond, Singh, Thomas 2022): The tropicalization of any graph profile is a rational polyhedral cone.

Acknowledgements

Both authors were partially supported by NSF grant DMS-2054404.

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