Variable Time Step Method of Dahlquist, Liniger, and Nevanlinna (DLN) for a Corrected Smagorinsky Model

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Abstract

Turbulent flows strain resources, both memory and CPU speed. DLN has greater accuracy and allows larger timesteps, requiring less memory and fewer FLOPS. DLN can also be implemented adaptively. The classical Smagorinsky model, as an effective way to approximate a (resolved) mean velocity, has recently been corrected to represent a flow of energy from unresolved fluctuations to the (resolved) mean velocity. In the report, we apply a family of second-order, G-stable time-stepping methods called DLN method to one corrected Smagorinsky model and provide the detailed numerical analysis about the stability and consistency. We prove that the numerical solutions under any arbitrary time step sequences are unconditional stable in long term and converge at second order. Numerical tests are given to confirm the rate of convergence and also to show that adaptive DLN helps to control numerical dissipation so that backscatter is visible.

Model Equations

- The CSM model reads $\nabla \cdot w = 0$ and
$$w_t - C_s^4\delta^2\mu^{-2}\nabla w_t + w\cdot\nabla w - \nu\nabla w + \nabla q - \nabla\cdot\Big((C_s\delta)^2|\nabla w|\nabla w\Big) = f.$$
In [1], the CSM model derivation and some basic properties of the CSM are developed and two algorithms (BE and CLNE) for its numerical simulation are proposed.
- μ is a constant from Kolmogorov-Prandtl relation, and $(w,q) \approx (\overline{u},\overline{p})$, $\nu_T = (C_s\delta)^2|\nabla w|$, $C_s \approx 0.1$, δ is a length scale.

Notations

- $X := \{v \in L^3(\Omega) : \nabla v \in L^3(\Omega) \text{ and } v = 0 \text{ on } \partial\Omega\}$, $Q := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q\,dx = 0\}$, and $V := \{v \in X : (q, \nabla\cdot v) = 0,\ \forall q \in Q\}$.
- $k_n = t_{n+1} - t_n$, $\epsilon_n = \frac{k_n - k_{n-1}}{k_n + k_{n-1}}$.
- $\alpha_2 = \frac{1}{2}(\theta+1)$, $\alpha_1 = -\theta$, $\alpha_0 = \frac{1}{2}(\theta-1)$, $\beta_2^{(n)} = \frac{1}{4}\Big(1 + \frac{1-\theta^2}{(1+\epsilon_n\theta)^2} + \epsilon_n^2\frac{\theta(1-\theta^2)}{(1+\epsilon_n\theta)^2} + \theta\Big)$, $\beta_1^{(n)} = \frac{1}{2}\Big(1 - \frac{1-\theta^2}{(1+\epsilon_n\theta)^2}\Big)$, $\beta_2^{(n)} = \frac{1}{4}\Big(1 + \frac{1-\theta^2}{(1+\epsilon_n\theta)^2} - \epsilon_n^2\frac{\theta(1-\theta^2)}{(1+\epsilon_n\theta)^2} - \theta\Big)$.
- $w_{n,\beta} = \beta_2^{(n)}w_{n+1} + \beta_1^{(n)}w_n + \beta_0^{(n)}w_{n-1}$, $\widehat{k_n} = \alpha_2 k_n - \alpha_0 k_{n-1} = \theta\frac{k_n - k_{n-1}}{2} + \frac{k_n + k_{n-1}}{2}$.
- $\widetilde{w_{n,\beta}^h} = \beta_2^{(n)}\Big\{\Big(1 + \frac{k_{n+1}}{k_n}\Big)w_n^h - \Big(\frac{k_{n+1}}{k_n}\Big)w_{n-1}^h\Big\} + \beta_1^{(n)}w_n^h + \beta_0^{(n)}w_{n-1}^h$.
- $a_1^{(n)} = -\frac{\sqrt{\theta(1-\theta^2)}}{\sqrt{2}(1+\epsilon_n\theta)}$, $a_2^{(n)} = -\frac{1-\epsilon_n}{2}a_1^{(n)}$, $a_0^{(n)} = -\frac{1+\epsilon_n}{2}a_1^{(n)}$.
- $$MD = \int_\Omega\Big(\frac{C_s^4\delta^2}{\mu^2}\frac{\alpha_2\nabla w_{n+1}^h + \alpha_1\nabla w_n^h + \alpha_0\nabla w_{n-1}^h}{\widehat{k_n}}\cdot\nabla w_{n,\beta}^h + (C_s\delta)^2|\nabla\widetilde{w_{n,\beta}^h}||\nabla w_{n,\beta}^h|^2\Big)\,dx.$$
$$CSMD = \int_\Omega\Big(\frac{C_s^4\delta^2}{\mu^2}\frac{\alpha_2\nabla w_{n+1}^h + \alpha_1\nabla w_n^h + \alpha_0\nabla w_{n-1}^h}{\widehat{k_n}}\cdot\nabla w_{n,\beta}^h\Big)\,dx.$$
$$ND = \Big\|\frac{\sum_{l=0}^2 a_l^n w_{n-1+l}^h}{\sqrt{\hat{k_n}}}\Big\|^2 + \frac{C_s^4\delta^2}{\mu^2}\Big\|\frac{\sum_{l=0}^2 a_l^n\nabla w_{n-1+l}^h}{\sqrt{\hat{k_n}}}\Big\|^2.$$
$$VD = \nu\|\nabla w_{n,\beta}^h\|^2, KE = \frac{1}{2}\|w_n^h\|^2.$$
- Minimum Dissipation criteria, $\chi = \left|\frac{\text{ND}}{\text{VD}}\right| < \text{Tol}$.

Full Discretization

Given $w_n^h,\ w_{n-1}^h \in X^h$ and $q_n^h,\ q_{n-1}^h \in Q^h$, find w_{n+1}^h and q_{n+1}^h satisfying
$$\Big(\frac{\alpha_2 w_{n+1}^h + \alpha_1 w_n^h + \alpha_0 w_{n-1}^h}{\widehat{k_n}}, v^h\Big) + \nu(\nabla w_{n,\beta}^h, \nabla v^h)$$
$$+ \frac{C_s^4\delta^2}{\mu^2}\Big(\frac{\alpha_2\nabla w_{n+1}^h + \alpha_1\nabla w_n^h + \alpha_0\nabla w_{n-1}^h}{\widehat{k_n}}, \nabla v^h\Big) + b^*(w_{n,\beta}^h, w_{n,\beta}^h, v^h)$$
$$- (q_{n,\beta}^h, \nabla\cdot v^h) + \Big((C_s\delta)^2|\nabla w_{n,\beta}^h|\nabla w_{n,\beta}^h, \nabla v^h\Big) = (f(t_{n,\beta}), v^h),\ \forall v^h \in X^h,$$
$$(\nabla\cdot w_{n,\beta}^h, p^h) = 0,\ \forall p^h \in Q^h.$$

Stability of DLN for CSM

The one-leg DLN method is unconditionally, long-time stable, i.e. for any integer $N > 1$,
$$\frac{1}{4}(1+\theta)(\|w_N^h\|^2 + \frac{C_s^4\delta^2}{\mu^2}\|\nabla w_N^h\|^2) + \frac{1}{4}(1-\theta)(\|w_{N-1}^h\|^2 + \frac{C_s^4\delta^2}{\mu^2}\|\nabla w_{N-1}^h\|^2)$$
$$+ \sum_{n=1}^{N-1}\Big(\Big\|\sum_{l=0}^{2} a_l^{(n)} w_{n-1+l}^h\Big\|^2 + \frac{C_s^4\delta^2}{\mu^2}\Big\|\sum_{l=0}^{2} a_l^{(n)}\nabla w_{n-1+l}^h\Big\|^2\Big)$$
$$+ \sum_{n=1}^{N-1}\widehat{k_n}\int_\Omega(\nu + (C_s\delta)^2|\nabla w_{n,\beta}^h|)|\nabla w_{n,\beta}^h|^2\,dx = \sum_{n=1}^{N-1}\widehat{k_n}(f_{n,\beta}, w_{n,\beta}^h)$$
$$+ \frac{1}{4}(1+\theta)(\|w_1^h\|^2 + \frac{C_s^4\delta^2}{\mu^2}\|\nabla w_1^h\|^2) + \frac{1}{4}(1-\theta)(\|w_0^h\|^2 + \frac{C_s^4\delta^2}{\mu^2}\|\nabla w_0^h\|^2).$$

Error Estimate

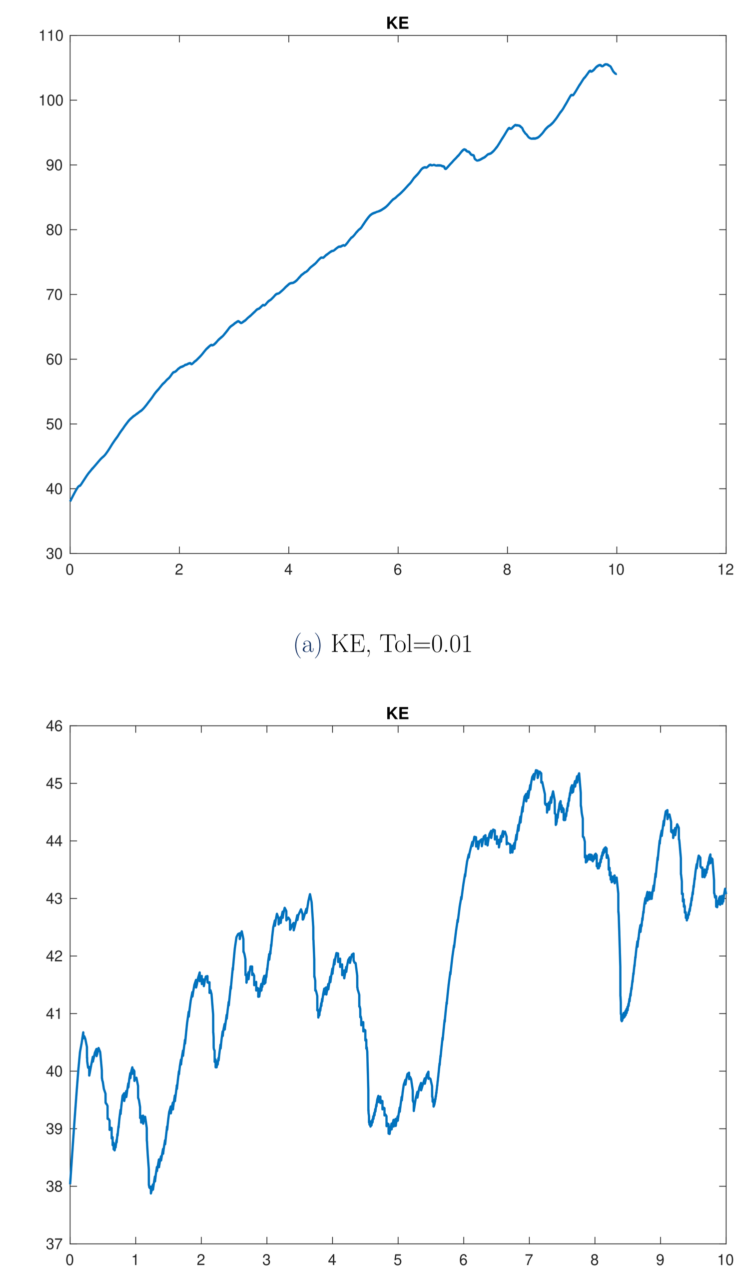
- Timestep condition:
$$\frac{C(\theta)k_{\max}}{\nu^3}\big(k_{\max}^6\|\nabla w_{tt}\|_{L^2(0,T;L^2)}^4 + C\|\nabla w\|_{L^\infty(0,T;L^2)}^4\big) < 1. \qquad (1)$$
- Let $(w(t), q(t))$ be sufficiently smooth, strong solution of the CSM. When applying one-leg DLN's algorithm, there is a constant $C > 0$ such that under timestep condition (1), the following error estimates hold
$$\|w_N - w_N^h\| + \frac{C_s^4\delta^2}{\mu^2}\|\nabla(w_N - w_N^h)\| \le \mathcal{O}\big(k_{\max}^2, h^2, \delta k_{\max}^{3/2}, \delta h^{3/2}\big).$$

Convergence Rate

Table 1: Errors by $\|\cdot\|_{\infty,0}$-norm and Convergence Rate for the constant DLN with $\theta = 2/3$

Timestep k	Meshsize h	$\|\|\|e^w\|\|\|_{\infty,0}$	Rate	$\|\|\|\nabla e^w\|\|\|_{\infty,0}$	Rate	$\|\|\|e^p\|\|\|_{\infty,0}$	Rate
0.08	0.09	6.03	-	56.85	-	10.86	-
0.04	0.042	0.049	6.92	1.36	5.39	0.08	7.10
0.02	0.02	0.01	2.06	0.39	1.76	0.02	2.04
0.01	0.01	0.003	2.009	0.12	1.94	0.005	1.98

KE vs. Time



(a) KE, Tol=0.01

(b) KE, Tol=0.05

Figure 1: Variable Step DLN with $Re = 10,000$, $\theta = 0.95$, $C_s = 0.1$, $\mu = 0.4$. (a): The pattern is visible when there is no backscatter and b): The pattern is visible when there is backscatter.

Important Result

The adaptive variable timestep DLN method applied to the Turbulent model (CSM) succeeds in showing backscatter while constant timestep fails.

MD and CSMD (Constant Timestep)

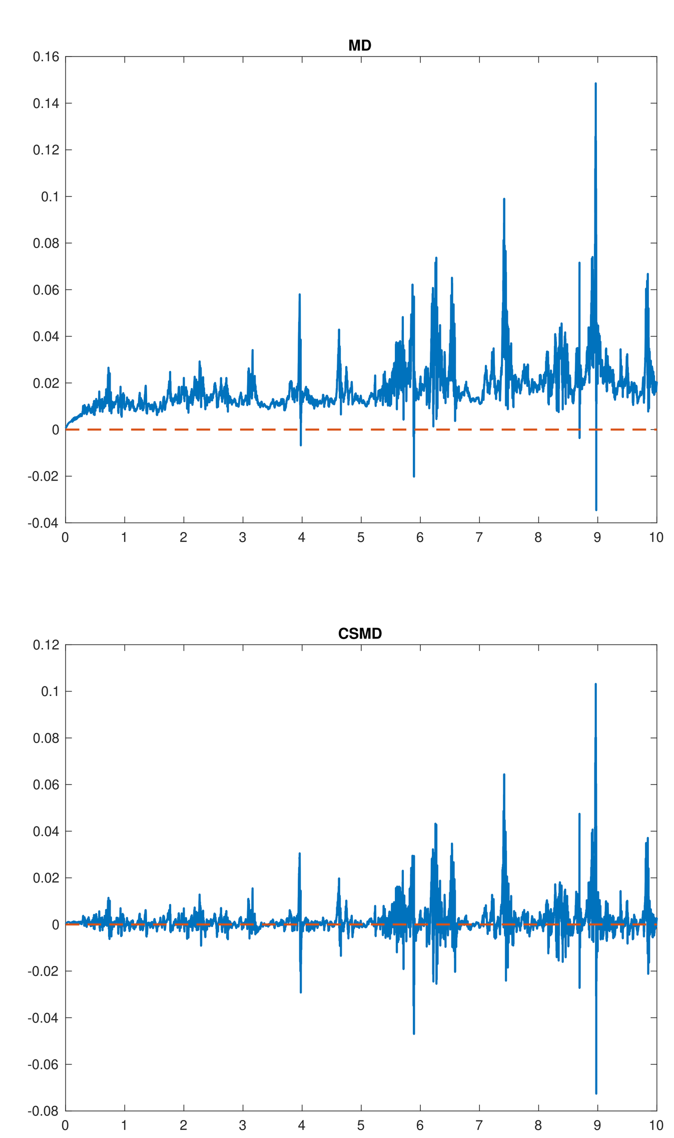


Figure 2: Constant DLN with $Re = 10,000$, $\theta = 0.95$, $C_s = 0.1$, $\mu = 0.4$.

MD and CSMD (Variable Timestep)

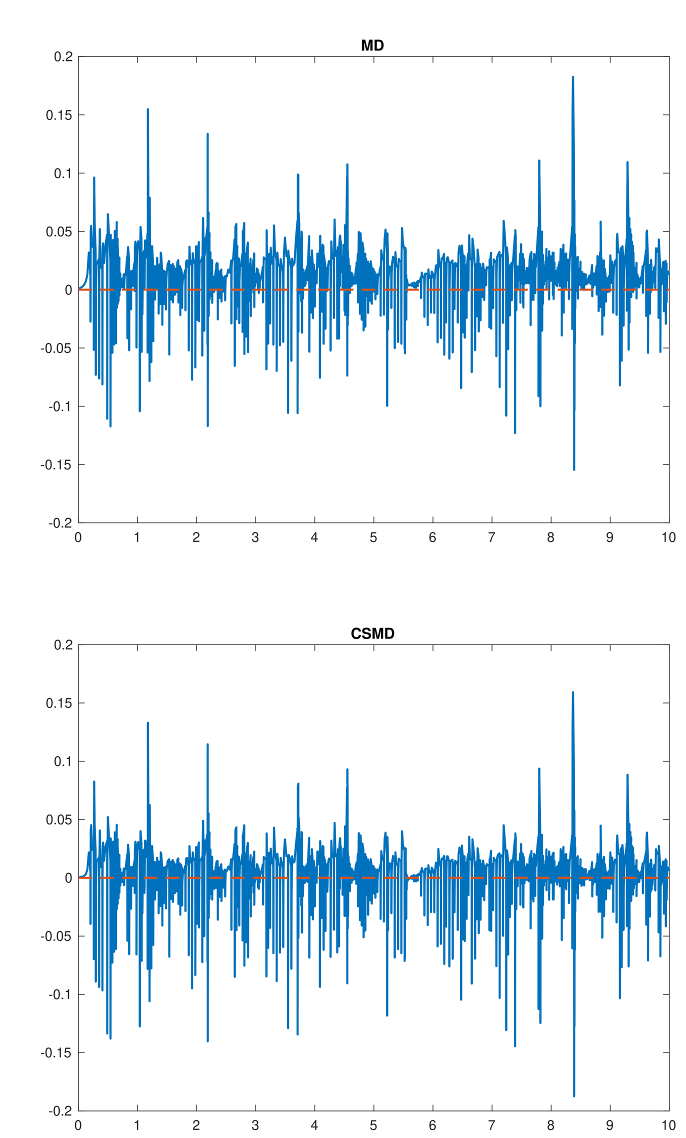
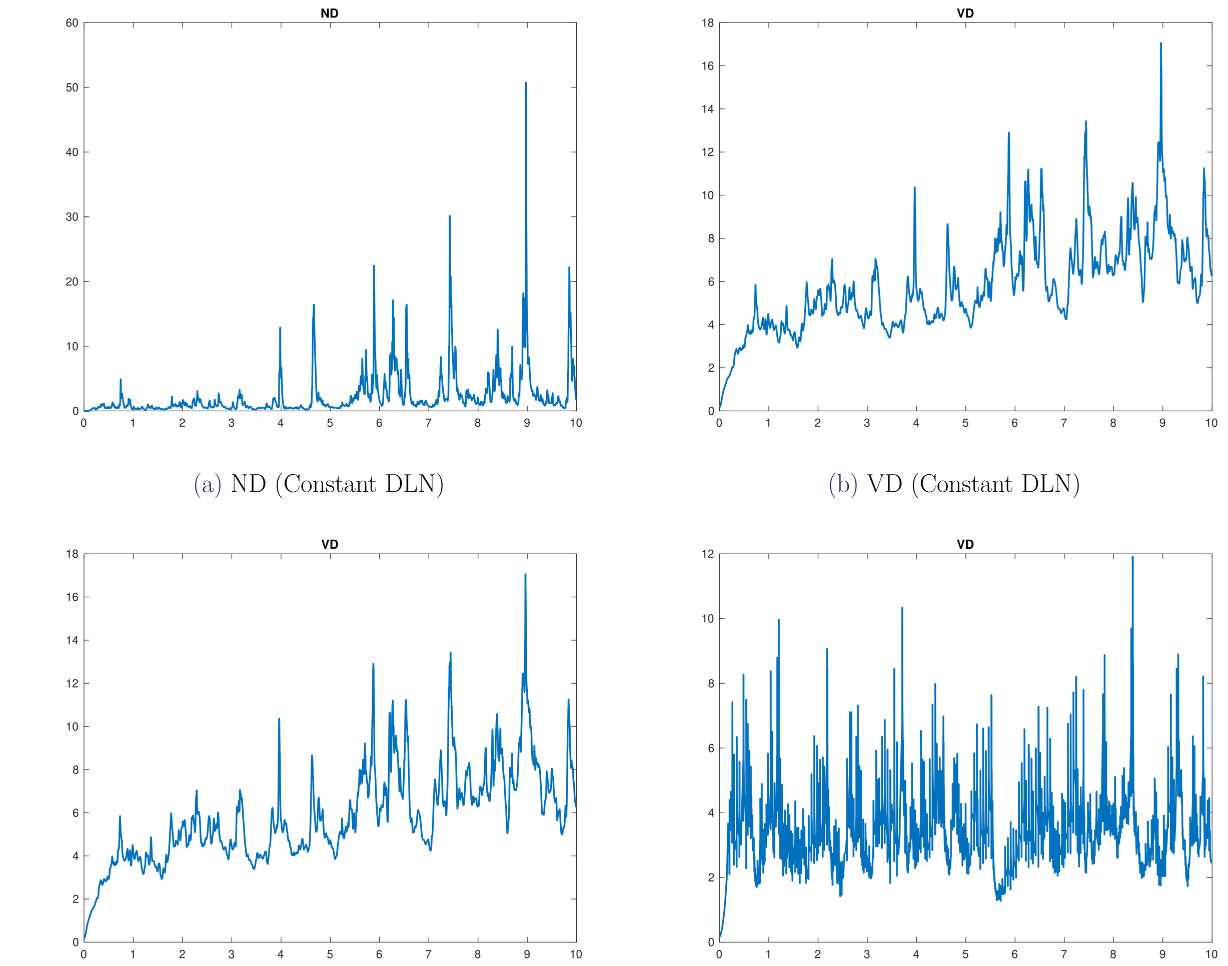


Figure 3: Adaptive DLN with $Tol = 0.05$, $Re = 10,000$, $\theta = 0.95$, $C_s = 0.1$ $\mu = 0.4$.

ND and VD



(a) ND (Constant DLN)

(b) VD (Constant DLN)

(c) ND (Adaptive DLN)

(d) VD (Adaptive DLN)

Figure 4: Adaptive DLN with $Tol = 0.05$, $Re = 10,000$, $\theta = 0.95$, $C_s = 0.1$ $\mu = 0.4$.

Total Timesteps

Table 2: Total timesteps taken to reach $T = 10$ while using variable DLN for different values of θ.

θ	Tol	Total Timesteps
0.98	**0.01**	**9575**
0.95	0.01	6505
0.95	**0.05**	**1604**
$2/\sqrt{5}$	0.01	8988
$2/\sqrt{5}$	0.05	5680
$\mathbf{2/\sqrt{5}}$	**0.15**	**1973**
$2/3$	0.01	9944
$2/3$	0.05	9575
$2/3$	0.15	7149

Conclusion and Future Work

The closer $\theta = 1$, the closer DLN method gets to be exactly conservative. If it is exactly conservative, we do not need tight control over ND. The further we go away from exactly conservative, the tighter control we need over ND to see bacscatter. Next, to avoid the timestep condition we will work on the linearly implicit DLN method.

References

[1] F. Siddiqua and X. Xie.
Numerical analysis of a corrected Smagorinsky model.
Numer. Methods Partial Differ. Eq., 39(1):356–382, 2023.

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