Phase and absorption contrast imaging using intensity measurements

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Support: AFOSR FA9550-21-1-0196

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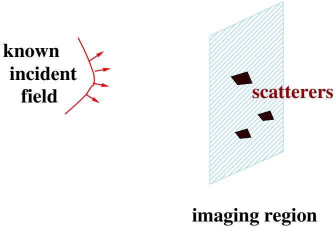
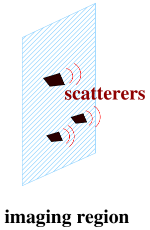
Inverse problems
Wave propagation



- Inverse problems aim to reconstruct a medium characteristics from knowledge of the response of the medium to a known incident field.
- In this talk we seek to reconstruct the transmisivity by recording the medium's response to one or more known excitations.

Inverse problems
Wave propagation



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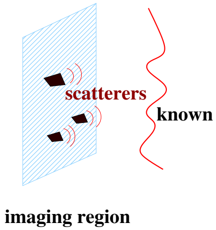

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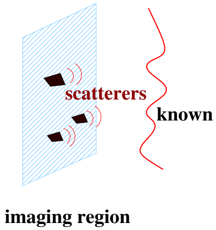

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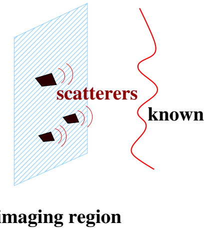

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- We consider a sparse unknown : the unknown image often has a low dimensional structure and admits a sparse representation in certain bases.
- Measurements : intensity-only.

Applications

At high frequencies intensities only can be recorded
e.g., CCD's, light detectors can record only intensities

- Optics
- Digital microscopy
- X-ray crystallography

We have developed a computational imaging approach that allows for phase and absorption contrast recovery from intensity measurements.

Multiple illuminations are needed (usual in phase retrieval ; masks).

The keystone for the efficiency of the method is a *robust dimensionality reduction* strategy carried in two steps accounting for both the incoherent (absorption contrast) and coherent contributions (phase contrast) in the data.

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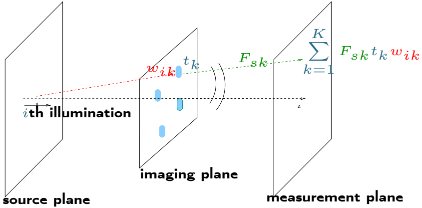
Imaging problem setup

We seek the transmissivity vector

$$\boldsymbol{t} = [t_1, \ldots, t_K]^\intercal = [|t_1|e^{i\varphi_1}, \ldots, |t_K|e^{i\varphi_K}]^\intercal \in \mathbb{C}^K$$

from intensity measurements of the form



$$|(\boldsymbol{b}_i)_s|^2 = \left| \sum_{k=1}^{K} F_{sk} w_{ik} t_k \right|^2$$
$$= \sum_{k=1}^{K} \underbrace{|F_{sk}|^2}_{=cste} |w_{ik}|^2 |t_k|^2 + \sum_{k=1}^{K} \sum_{\substack{k'=1 \\ k' \neq k}}^{K} F_{sk} F^*_{sk'} w_{ik} w^*_{ik'} t_k t^*_{k'}$$

$|(\boldsymbol{b}_i)_s|^2$ is the intensity recorded at the s-th transducer when the ith illumination

$$\boldsymbol{w}_i = [w_{i1}, \ldots, w_{iK}]^\intercal \in \mathbb{C}^K$$

impinges on the object plane. F_{sk} is the propagator from the object plane to the receiver plane. F and $\boldsymbol{w}_i$ are assumed known.

Imaging problem setup

This problem can be written in matrix form as

$$\mathcal{W}_{incoh}\ \boldsymbol{\chi}_d + \mathcal{W}_{coh}\ \boldsymbol{\chi}_{cross} = \boldsymbol{d}$$

The data are

$$\boldsymbol{d} = [\boldsymbol{d}_1^T, \boldsymbol{d}_2^T, \ldots, \boldsymbol{d}_S^T]^T$$

with $\boldsymbol{d}_s = [|(\boldsymbol{b}_1)_s|^2, |(\boldsymbol{b}_2)_s|^2, \ldots, |(\boldsymbol{b}_N)_s|^2]^T$ the intensities recorded at the detector s for the illuminations $1, 2, \ldots, N$.

The unknown is decomposed into

$$\boldsymbol{\chi}_d = [|t_1|^2, |t_2|^2, \ldots, |t_K|^2]^T$$

and

$$\boldsymbol{\chi}_{cross} = [t_1 t_2^*, t_1 t_3^*, \ldots, t_1 t_K^*, t_2 t_1^*, t_2 t_3^*, \ldots, t_2 t_K^*, t_3 t_1^*, \ldots,]\,,$$

The bottleneck for the inversion is the size of the problem, which is enormous if one wants to form high resolution images. An image with 1000×1000 pixels, amounts to solving a linear system with 10^{12} unknowns!

Imaging problem setup

The problem of recovering t from intensity measurements is nonlinear and there is much interest in finding algorithms that give the true global solution effectively.

Iterative projection methods

R.W. Gerchberg and W.O. Saxton, *A practical algorithm for the determination of phase from image and diffraction plane pictures*, Optik 35, 237-246 (1972).

J.R. Fienup, *Reconstruction of an object from the modulus of its Fourier transform*, Optics Letters 3, 27-29 (1978).

simple to implement & very flexible in practice

do not always converge to the true solution unless good prior information is available.

Imaging problem setup

The problem of recovering t from intensity measurements is nonlinear and there is much interest in finding algorithms that give the true global solution effectively.

Quadratic methods seek for the matrix unknown tt^* using nuclear norm minimization

A. Chai, M. Moscoso and G. Papanicolaou, *Array imaging using intensity-only measurements*, Inverse Problems 27 (2011), 015005.

E. J. Candès, Y. C. Eldar, T. Strohmer, and V. Voroninski, *Phase Retrieval via Matrix Completion*, SIAM J. on Imaging Sci. 6 (2013), 199-225.

convex problem $\rightsquigarrow$ convergence to the true solution

computational complexity limits the usefulness of this approach

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The noise collector and dimension reduction

We propose the following robust dimensionality reduction strategy.

Instead of solving the problem with K^2 unknowns we reduce its dimensionality constructing linear problems for only $O(K)$ unknowns & absorb the error, that is the *contribution of the unmodeled unknowns* using a *Noise Collector*.

We solve linear systems of the form

$$\mathcal{A}\boldsymbol{\chi} + \mathcal{C}\boldsymbol{\eta} = \boldsymbol{d}$$

- $\mathcal{A}$ a matrix with $O(K)$ subsampled columns of $[\mathcal{W}_{incoh}|\mathcal{W}_{coh}]$.
- $\boldsymbol{\chi}$ is a sparse vector that represents the object
- $\boldsymbol{\eta}$ is an auxiliary unknown introduced to absorb the error
- $\mathcal{C}$ is a *Noise Collector* matrix.
- This approach allows us to find the exact support of χ for each linear problem we solve (incoherent & coherent)

The Noise collector

The Noise Collector is a method that allows us to find the sparse solution $\boldsymbol{\chi} \in \mathbb{C}^{\mathcal{K}}$ of

$$\mathcal{A}\boldsymbol{\chi} = \boldsymbol{d}(= \boldsymbol{d}_0 + \boldsymbol{e})$$

from highly incomplete ($1 \ll N < K$) and noisy data $\boldsymbol{d} \in \mathbb{C}^N$ (noise $\boldsymbol{e} \in \mathbb{C}^N$).

Main result : The support of $\boldsymbol{\chi}_\tau$ found as

$$(\boldsymbol{\chi}_\tau, \boldsymbol{\eta}_\tau) = \arg\min_{\boldsymbol{\chi},\boldsymbol{\eta}} \left(\tau\|\boldsymbol{\chi}\|_{\ell_1} + \|\boldsymbol{\eta}\|_{\ell_1}\right), \quad \text{subject to } \mathcal{A}\boldsymbol{\chi} + \mathcal{C}\boldsymbol{\eta} = \boldsymbol{d}$$

is exact when the noise is not too large.

$\mathcal{C}$ is the *Noise Collector* matrix $\mathcal{C} \in \mathbb{C}^{N\times\Sigma}$, $\Sigma = N^\beta$, for $\beta > 1$ and τ is an $O(1)$ no-phantom weight that is independent of the dimension of the problem and the level of noise in the data.

η does not correspond to a physical quantity. It is introduced to provide an appropriate linear combination of the columns of $\mathcal{C}$ that produces a good approximation to the noise vector e. $\mathcal{C}\boldsymbol{\eta}_\tau$ absorbs *all* the noise (and possibly some signal).

The Noise collector

- The columns of $\mathcal{C}$ are chosen independently and at random on the unit sphere $\mathbb{S}^{N-1}$ so that we could approximate well a typical noise vector.

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- The weight $\tau > 1$ is chosen so it is expensive to approximate $\boldsymbol{e}$ with the columns of $\mathcal{A}$. τ cannot be taken too large because then the collector becomes too "cheap" and we lose the signal $\boldsymbol{\chi}$ that gets also absorbed by the *Noise Collector*. In practice, it is chosen as the minimal τ so that $\boldsymbol{\chi} = 0$ when $\boldsymbol{d} = \boldsymbol{e}$ (pure noise data) - no-phantom weight.

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- The main result is obtained under the assumption that the columns of $\mathcal{A}$ are incoherent,

$$|\langle \boldsymbol{a}_i, \boldsymbol{a}_j \rangle| \leqslant \frac{1}{3M} \text{ for all } i \text{ and } j,$$

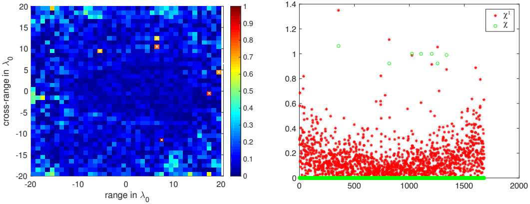
and that the noise is not too large

$$\max\left(1, \|\boldsymbol{e}\|_{\ell_2}\right) \leqslant c_1 \frac{\|\boldsymbol{d}_0\|_{\ell_2}^2}{\|\boldsymbol{\chi}\|_{\ell_1}} \sqrt{\frac{N}{\ln N}},$$

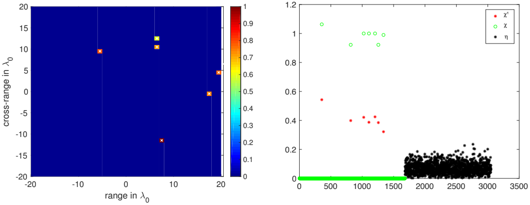
The Noise collector

Noise Collector at work

ℓ_1 reconstruction without the Noise Collector



ℓ_1 reconstruction with the Noise Collector



$N = 1369$ measurements. $K = 1681$ pixels in the images. 100% noise.

The Noise Collector allows for exact support recovery!

The algorithm

The algorithm has three steps

(1) In the first step, we seek the strong absorbing objects. We set $\mathcal{A} = \mathcal{W}_{incoh}$, and solve

$$\mathcal{A}\boldsymbol{\chi} + \mathcal{C}\boldsymbol{\eta} = \boldsymbol{d} \, (= \mathcal{W}_{incoh}\boldsymbol{\chi}_d + \mathcal{W}_{coh}\boldsymbol{\chi}_{cross}),$$

for $\boldsymbol{\chi} = \boldsymbol{\chi}_d = [|t_1|^2, |t_2|^2, \ldots, |t_K|^2]^T$.

$\mathcal{C}\boldsymbol{\eta}$ absorbs the contributions of χ_{cross} to the intensities which are treated in this step as noise. The model is not exact so only the strong absorbers are detected.

The first term in $|(\boldsymbol{b}_i)_s|^2 = \underbrace{\sum_{k=1}^{K} |w_{ik}|^2 |t_k|^2}_{\text{indep of s}} + \sum_{k=1}^{m} \sum_{\substack{k'=1 \\ k' \neq k}}^{K} F_{sk} F^*_{sk'} w_{ik} w^*_{ik'} t_k t^*_{k'}$

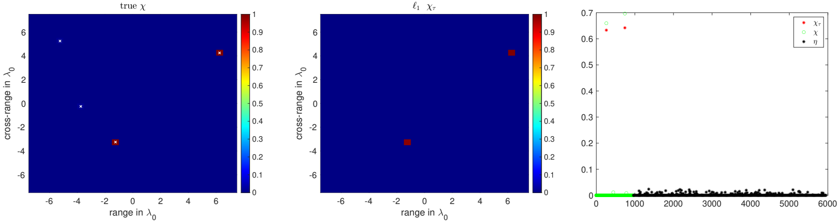
is independent of s $\Rightarrow$ use total intensity as data ; no need to know the propagator F_{sk}.

Consider m *strong absorbers* $|t_i| = O(1)$, $i = 1, \ldots, m$ *and* n *weak (phase contrast)* $|t_j| = O(\varepsilon)$, $j = 1, \ldots, n$. *During the first step we only recover* $|t_i|^2$, $i = 1, \ldots, m$ *because the contribution from* $|t_j|^2 = O(\varepsilon^2)$ $j = 1, \ldots, n$ *is lost in the noise.*

The algorithm

Example. Imaging two strong (red squares) and two weak (white crosses) absorbers $m = 2$, $n = 2$.

true χ_d reconstructed χ_d



First step : Recovering the two strong ones. The total power received for $N = 300$ illumination patterns is used as data. The unknown dimension is $K = 961$ $(K^2 = 923521)$

The algorithm

(2) In the second step :

- We first remove from the data the contributions already found ($O(1)$ *contributions*) what remains is

$$|(\boldsymbol{b}_i)_s|^2 = \underbrace{\sum_{k=1}^{n} |w_{ik}|^2 |t_k|^2}_{O(\varepsilon^2)} + \underbrace{\sum_{k=1}^{m} \sum_{\substack{k'=1 \\ k' \neq k}}^{m} F_{sk} F^*_{sk'} w_{ik} w^*_{ik'} t_k t^*_{k'}}_{O(1)}$$

$$+ \underbrace{\sum_{k=1}^{m} \sum_{\substack{k'=1 \\ k' \neq k}}^{n} F_{sk} F^*_{sk'} w_{ik} w^*_{ik'} t_k t^*_{k'}}_{O(\varepsilon)} + \underbrace{\sum_{k=1}^{n} \sum_{\substack{k'=1 \\ k' \neq k}}^{n} F_{sk} F^*_{sk'} w_{ik} w^*_{ik'} t_k t^*_{k'}}_{O(\varepsilon^2)}$$

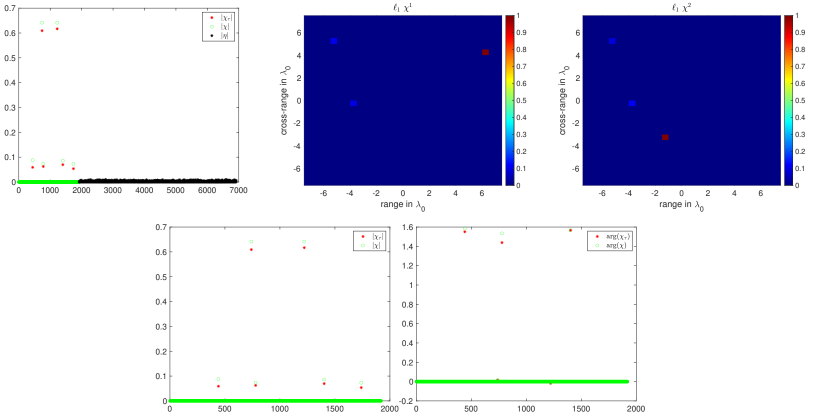
- Then for every pixel $i = 1, \ldots, m$ detected during the first step we seek for its interactions $t_i^* t_j$ with all the other $K - 1$ pixels in the object plane, $j = 1, \ldots, K$, $j \neq i$ ($O(1)$ *and* $O(\varepsilon)$ *contributions*).

In this case $\mathcal{A} = (\mathcal{W}_{coh})_{sub}$, where $(\mathcal{W}_{coh})_{sub}$ contains the $m\,(K-1)$ columns that correspond to the interactions between the m detected objects in the first step and all the other pixels in the image.

Since we are neglecting the $O(\varepsilon^2)$ *contributions*, the system is not exact.

The algorithm

Example. Imaging two strong (red squares) and two weak (white crosses) absorbers $m = 2$, $n = 2$.



Second step : Recovering the two weak ones. The power received on 5×5 receivers for $N = 300$ illumination patterns is used as data. The unknown dimension is $2(K-1) = 1920$ ($K^2 = 923521$)

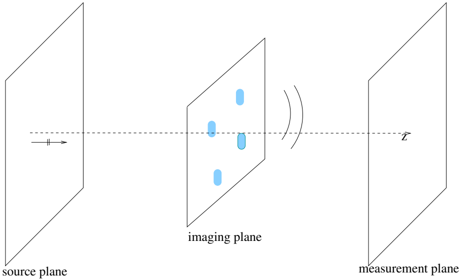
The algorithm

(3) The third step is optional. It is used to obtain more precise quantitative images.

Once the strong and weak absorbing objects are found, we solve the full problem but restricted to the recovered support.

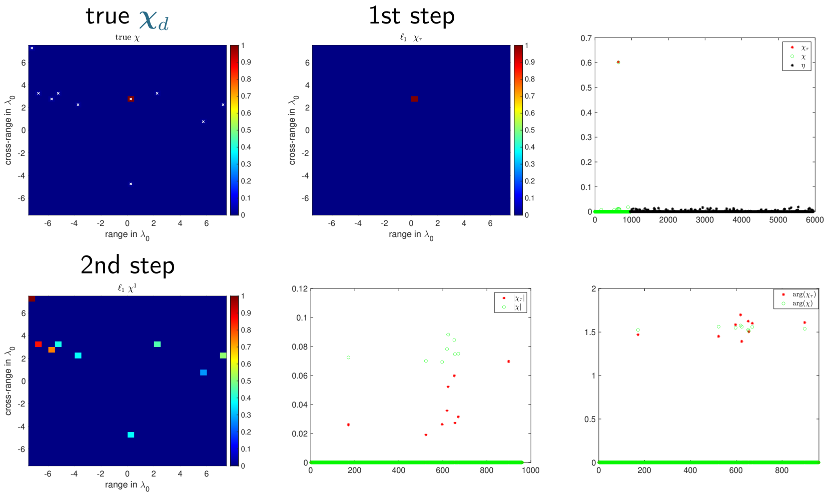
This is now a small problem that can be solved using an ℓ_2 minimization method that gives very accurate results.

Setup : transmission problem with multiple illuminations



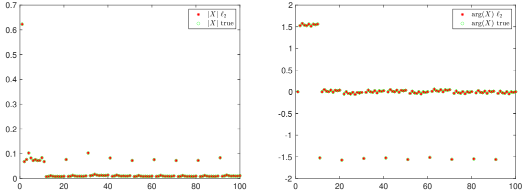
- Wavelength $\lambda = 500$ nm
- Source plane : 21×21 evenly distributed sources on 8mm×8mm at $z = -8$mm. (8mm = 16000 λ)
- $N = 300$ different illumination patterns are used.
- Imaging region 31×31 pixels centered at the origin. Thin object. pixel size $\lambda/2 = 250$nm.
- Measurements sampled on 5×5 receivers located on a 8mm×8mm aperture at $z = +8$mm.

Results (1 strong and 9 weak absorbers)



First step and second steps for 1 strong and 9 weak absorbers ($m = 1$, $n = 9$)
SNR$= 30$dB.

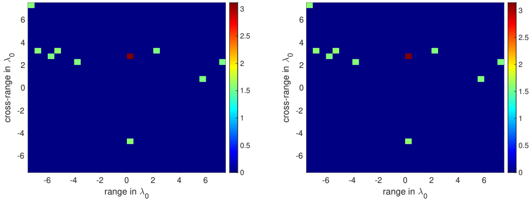
Results (1 strong and 9 weak absorbers)



Third step for the full unknown $X = tt^*$ restricted to the recovered support

The dimension of the unknown is 10^2.

true SNR=30dB



True and recovered phase maps for the 10 absorbers.

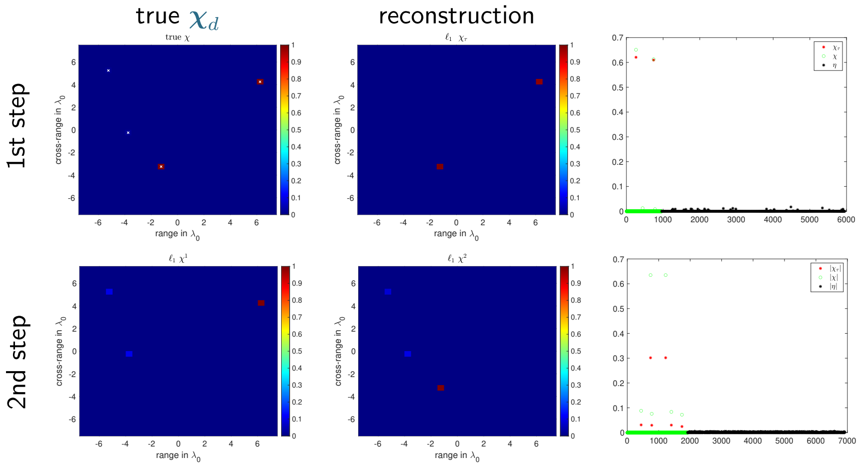
Partially coherent data

We use the following model to generate the data

$$|(\boldsymbol{b}_i)_s|^2 = \sum_{k=1}^{K} |w_{ik}|^2 |t_k|^2 + \alpha_{coh} \sum_{k=1}^{K} \sum_{\substack{k'=1 \\ k' \neq k}}^{K} F_{sk} F^*_{sk'} w_{ik} w^*_{ik'} t_k t^*_{k'} \,,$$

with $0 \leq \alpha_{coh} \leq 1$. If $\alpha_{coh} = 1$, the sources are fully coherent, and if $\alpha_{coh} = 0$ they are fully incoherent. This parameter is unknown for the inversion of the data.

Partially coherent data



Imaging 2 strong and 2 weak absorbers with partially coherent illumination.

Here $\alpha_{coh} = 0.5$. As α_{coh} decreases we may lose the weak absorbers. This depends on the transparency of these objects, their number, and the noise in the data.

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Concluding remarks

- We presented a two (*three*) step algorithm for phase retrieval based on a *robust dimensionality reduction* strategy carried in two steps accounting for both the incoherent (absorption contrast) and coherent contributions (phase contrast) in the data.
- The algorithm is efficient because its cost is linear in the number of pixels !
- It guarantees exact recovery if the image is sparse with respect to a given basis.
- May be used, without any modification, for partially coherent data. This is very important for phase-contrast X-ray imaging because fully coherent sources of X-rays are very hard to be obtained.

Concluding remarks

- More on the Noise Collector and its theoretical analysis in

M. Moscoso, A. Novikov, G. Papanicolaou, CT, *Imaging with highly incomplete and corrupted data*, Inverse Problems, 36(3), p. 035010, 2020. https://doi.org/10.1088/1361-6420/ab5a21

M. Moscoso, A. Novikov, G. Papanicolaou, CT, *The Noise Collector for sparse recovery in high dimensions*, Proceedings of the National Academy of Sciences, 117 (21), p. 11226-11232, 2020. https://doi.org/10.1073/pnas.1913995117

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- More on the Noise Collector for quadratic (cross-correlation) measurements

M. Moscoso, A. Novikov, G. Papanicolaou, CT, *Fast signal recovery from quadratic measurements*, IEEE Transactions on Signal Processing, vol. 69, pp. 2042–2055, 2021. doi:10.1109/TSP.2021.3067140 (deterministic case)

M. Moscoso, A. Novikov, G. Papanicolaou, CT, *Quantitative phase and absorption contrast imaging*, IEEE Transactions on Computational Imaging, vol. 8, pp. 784-794, 2022. doi:10.1109/TCI.2022.3204401.

M. Moscoso, A. Novikov, G. Papanicolaou, CT, *The random case : coming soon*