RESEARCH STATEMENT – ROSEMARIE BONGERS

My research lies at the intersection of geometric measure theory, quasiconformal maps, and complex and harmonic analysis. In many mathematical contexts, we can either estimate a quantity or constrain the structure of the examples which do not follow a given bound. In my work, this idea motivates the use of geometric measure theory techniques (and the structural properties of fractal sets) to build and study extremizers in geometric function theory.

The first area of my work is on **extremal quasiconformal maps** through the lens of regularity. Quasiconformal maps generalize the idea of conformality by relaxing the geometry; while a conformal map takes infinitesimal circles to infinitesimal circles, a quasiconformal map can distort space in a bounded way without tearing.

- (§1.1) In [14], I used techniques from geometric measure theory to construct examples of quasiconformal maps which exhibit irregular stretching and rotation behavior on very large sets; this involved the construction of highly **non-self-similar fractal sets** derived from Cantor sets, and gave an unexpectedly strong lower bound for the sizes of the sets.
- (§1.2) The local structure of the partial differential equation underlying a quasiconformal map can be used to **improve regularity estimates** for the solution. In [13], I presented improved Hölder continuity results for a broad class of quasiconformal maps, and quantified this improvement in terms of the local geometry of the map. This work relied on tools from elliptic PDEs.
- (§1.3) In [15], we gave an explicit construction of (real) quasiconformal maps in all dimensions which exhibit the worst-case stretching behavior on any prescribed countable set, relying on delicate **linear algebraic estimates** of quantities measuring geometric distortion.

A second area of my work is in geometric measure theory. Following the theme of using fractal sets to construct **extremizers in geometric function theory**, I also study the average projection length of sets in Euclidean spaces; this is closely related to classifying sets via rectifiability.

- (§2.1) In [12], I found new geometrically motivated techniques to connect the **Hausdorff dimension** of a set to the **decay rate** of the projected lengths of its neighborhoods. These techniques generalized to a large class of self-similar fractal sets in the plane which arise naturally in connection with complex analysis and removability problems.
- (§2.2) Potential theory methods can be used to study the geometric arrangement and size of a set via its orthogonal projections. In [17], we used a geometrically motivated **transversality condition** to extend these methods to a broad class of non-linear projection-type maps.

A third area of work is in **singular integrals and commutators**; such operators arise in many areas of analysis (e.g. the Beurling transform's connection to quasiconformal geometry). Many classical results in harmonic analysis arise from the study of cancellation in integral operators. One particular place where this idea arises naturally is in the connection between singular integral operators and commutators, where oscillation and cancellation play a subtle role and combine to give strong boundedness results.

- (§3.1) In [16], we developed new techniques to study commutators associated to a singular integral operator based on a monomial curve; this has a much worse singularity than classically studied operators such as the Hilbert and Riesz transforms, and there are numerous technical difficulties that arise in characterizing the commutators which are bounded on various function spaces. The questions that arise in this field are closely related to **one- and two-weight inequalities for singular integrals**.

The following sections describe these results in detail. Directions for **current and future work** are described in §1.4, §2.3–4, and §3.2–3. Work that could involve **undergraduate** researchers is described in §2.3.

1. Geometry and Distortion of Quasiconformal Maps

A K-*quasiconformal* (K-QC) map for $K \geq 1$ is an orientation preserving homeomorphism (with locally L^2 derivative) between two regions in the complex plane that satisfies particular distortion inequalities; infinitesimally, circles are mapped to ellipses whose eccentricities are uniformly bounded in terms of K. These maps provide an important and flexible generalization of conformal maps, and have important connections with complex dynamics and elliptic PDEs, as well as problems in elasticity and fluid dynamics. Regularity of these mappings has been studied in many papers, such as the Hölder regularity results found by Mori [32] and the Salem Prize-winning area distortion work of Astala [3]. Quasiconformal maps are also intimately related with the behavior of the Beurling transform and related Calderon-Zygmund operators; for example, weighted bounds for the Beurling transform were used by Lacey, Sawyer, and Uriarte-Tuero in [28] to control the distortion of Hausdorff dimension.

1.1. **Geometric distortion properties.**

In my research, I considered the multifractal spectrum for the simultaneous stretching and rotation set of a K-QC map. We say that a K-QC mapping stretches like α and rotates like γ at a point $z \in \mathbb{C}$ if there is a sequence of scales $r_n \to 0$ for which

$$\lim_{n\to\infty} \frac{\log|f(z+r_n)-f(z)|}{\log r_n} = \alpha \quad \text{and} \quad \lim_{n\to\infty} \frac{\arg|f(z+r_n)-f(z)|}{\log r_n} = \gamma \tag{1}$$

for an appropriate definition of the argument; this gives a quantitative local generalization of Hölder continuity at a point. Astala, Iwaniec, Prause and Saksman gave a precise bound on the Hausdorff dimension of the simultaneous stretching and rotation sets for given parameters α, γ and K in [4]; they showed that this was sharp at the level of dimension. Hitruhin [24] gave examples with positive but finite Hausdorff measure at the critical dimension.

By analogy with the techniques in area distortion, it was expected that this was sharp. Surprisingly, however, this breaks down; I improved and generalized Hitruhin's result to find a broad class of Hausdorff gauge functions for which the stretching and rotation set can have positive measure in [14]. Such gauged measures give a measure of the size of a set with a given Hausdorff dimension, but whose size is too large for the usual Hausdorff measure. In particular, I can improve the gauge function in [24] by more than a logarithmic factor; as a particular application, the sets can have non-σ-finite Hausdorff measure at the appropriate dimension. Moreover, the sets considered can have positive Riesz capacity $\dot{C}_{\beta,q}$ for all parameters β, q with the correct homogeneity (which is closely related to the work in [1]). Furthermore, in dimension zero, I showed that not only are there uncountable stretching and rotation sets, but that any bounded countable set E has an associated K-QC map that stretches in the worst possible fashion at every point in E.

The constructions used here involve non-self-similar Cantor type sets which originated in [39]. These sets are approximated by families of disks, and K-QC maps are defined which are adapted to the disks with prescribed stretching and rotation behavior. There are substantial technical difficulties in adapting the construction to increase the sizes of the Cantor sets without significantly disrupting the stretching and rotation behavior of the associated K-QC map.

1.2. **Hölder regularity.**

A K-QC mapping is necessarily $\frac{1}{K}$-Hölder continuous, and this is sharp due to the example of radial stretches $z|z|^{1/K-1}$. On the other hand, K-QC maps have useful self-improvement properties. Although they are *a priori* assumed to lie in the Sobolev class $W^{1,2}_{\text{loc}}$, it turns out the derivative is integrable up to exponent $2K/(K-1) > 2$ and similarly that the Jacobian $J_f \in L^{\frac{K}{K-1},\infty}$. This improvement is proved via holomorphic motions and with the techniques leading to Astala's area distortion theorem ([5], [3]). Since length distortion and area distortion are intimately connected for K-QC maps (because morally, a quasiconformal map distorts all directions in a comparable way), integrability of the Jacobian (which controls area distortion) leads directly to Hölder regularity of the map. In particular, any improvement in integrability past the critical exponent leads to better than expected Hölder exponent (although the converse is not true, even for maps arbitrarily close to bilipschitz [20]).

There is a broad class of K-QC mappings for which the Hölder exponent is better than one expects from the parameter K; for example, a K-quasiconformal map may be bilipschitz. The work of Ricciardi ([35], [36]) shows that the structure of the Beltrami coefficient can lead to increased regularity. In [13], I applied the isoperimetric inequality and local structure of the Beltrami equation to improve this regularity substantially.

The key ingredient is a completely explicit computation of the length of quasicircles (that is, quasiconformal images of circles) in terms of the Beltrami coefficient μ and the Jacobian J_f of the underlying QC map f: if S_t is a circle of radius t centered at the origin,

$$\mathcal{H}^1\big(f(S_t)\big) = \int_0^{2\pi} \left(\frac{|1 - e^{-2it}\mu|^2}{1 - |\mu|^2} \right)^{1/2} J_f^{1/2} t \, d\theta.$$

These results are also able to give constraints on the form of the Beltrami coefficient and the geometric behavior of the map when the associated solution has the worst-case Hölder continuity – roughly speaking, an extremizer for Hölder continuity must map circles to circles (in an appropriate quantitative sense) and have Beltrami coefficient similar to a radial stretch. In a more general context, I am working to find a translation between the structure of the Beltrami coefficient and the geometric behavior and regularity of the map. There are deep open problems concerning the recovery of Lipschitz or Hölder continuity from the Beltrami coefficient, and this is related to the connectivity of the manifold of chord-arc curves (e.g. [2] and Calderón's 1978 ICM address [18]).

1.3. **Extremizers in higher dimensions.** Quasiconformal maps can be studied in dimension greater than 2 as well; infinitesmally, balls are mapped to ellipsoids and this can be quantified in terms of the Jacobian determinant of the map. Many of the same regularity and geometric questions can still be analyzed, although without the tools from complex analysis specialized to the plane. K-QC maps in higher dimensions are still $\frac{1}{K}$-Hölder continuous, and a natural avenue of study is to characterize the sets on which a map can achieve its worst-case stretching behavior.

In [15], we constructed a K-QC extremizer which attains the worst-case stretching behavior on any given bounded, countable set in $\mathbb{R}^d$. The core of the construction is to sum radial stretches F_λ centered at each point in the given set Λ, taking

$$F(\vec{x}) = \sum_{\lambda \in \Lambda} F_\lambda(\vec{x}) := \sum_{\lambda \in \Lambda} \omega_\lambda(\vec{x} - \lambda) \|\vec{x} - \lambda\|^{1/K - 1}.$$

One must then show that the sum is both quasiconformal and admits the worst-case stretch. This is delicate for two reasons: first, the sum of quasiconformal maps is rarely quasiconformal (nor even a homeomorphism). Secondly, stretching is quantified through the Jacobian

$$J_F = \det(DF) = \det\left(\sum_{\lambda \in \Lambda} DF_\lambda \right)$$

which is a highly non-linear object. The key ingredient is a certain notion of positivity embedded in the derivative of a radial stretch that can be used to estimate the determinant of a sum of derivative matrices, paired with an estimate for the determinant of DF_λ arising from the structure of symmetric polynomials.

1.4. **Future work.** Quasiconformal maps in the plane have a particularly robust regularity theory in large part due to their connections (via the Beltrami equation) to more general divergence-type elliptic partial differential equations. One can ask how regularity extends to higher dimensions, in which case the Beltrami equation is no longer available; there are substantial difficulties that arise due to the different nature of quasiconformality in higher, real dimensions. However, length-area estimates related to the isoperimetric inequality are still highly relevant, and one can generalize some aspects of the planar approach. I am working on several aspects of this; in particular, studying the geometric behavior of extremizers for Hölder regularity has substantial overlap with the planar version (e.g. balls must map to balls with a quantitative bound on the geometric distortion). This gives a partial characterization of the extremizers, but we still have the following more general question:

Question. What can be said about the structure of the Jacobian of an extremizer for Hölder continuity in settings beyond quasiconformal maps in $\mathbb{C}$?

In the plane, this can be answered through the Beltrami coefficient; in higher dimensions, a new approach is needed and only partial results are known. A further generalization of this question applies to mappings of *finite distortion*; in this context, one requires that infinitesimal circles map to infinitesimal ellipses but without a uniform bound on eccentricity. Many of the same approaches (in terms of a geometric estimate and an analytic estimate involving a partial differential equation or inequality) show promise, and we are working to understand the extremizers for regularity theory in the spirit of the known sharp modulus of continuity results of [34].

2. Geometric measure theory, projections, and energy techniques

Given a set E in the plane, its Favard length is the average

$$\mathrm{Fav}(E) = \int_0^{2\pi} |\pi_\theta E| \, d\theta$$

where π_θ is projection onto a line through the origin at angle θ. This measures the average length of the projections of E, and is related to both the integralgeometric measure and Buffon needle probability of the set. This quantity is deeply related to the rectifiability of the set [9]; additionally, the decay rate of $\mathrm{Fav}(E(r))$ as r decreases carries information about the dimension and geometry of the set. Here $E(r)$ is the r-neighborhood of E, the set of points of distance at most r from E.

Favard length is intimately connected with the geometry of a set: a one dimensional set is purely unrectifiable if and only if it has Favard length zero. Vitushkin conjectured that pure unrectifiability is equivalent to zero analytic capacity, as was proved by David in [22] for sets of finite length. In general, we wish to understand the relationship between the size of a set, the Favard length, and the analytic capacity. It is still an open fundamental problem whether Favard length is always controlled by analytic capacity.

2.1. Asymptotic decay of Favard length.

The Favard length of an r-neighborhood of a purely unrectifiable set tends to zero as r does, and a connection between the decay asymptotics of Favard length and the size of the set was found in [30]. Lower bounds on the Favard length were given in terms of the Hausdorff dimension and measure of the underlying set, using potentials. I gave a new argument for this bound (at the level of dimension) based on a more direct geometric approach that avoids the use of Riesz potentials in [12]. My approach also relaxes some of the topological requirements on the set.

I have given several new geometrically motivated arguments for lower bounds on the decay of Favard length of one dimensional sets. In particular, I generalized the square-counting technique of [6] to give a logarithmic lower bound for unions of disks whose projections have bounded overlap in at least one direction (which we refer to as a *critical angle*). Furthermore, I have developed a new technique for studying self-similar sets with a critical angle: the sequence of Favard lengths when moving between different generations of the set is convex, which leads directly to analytically useful lower bounds on the lengths.

2.2. Projections and transversality.

A key tool in projection theory is based on the estimate

$$\mathrm{Fav}(E) \gtrsim I_1(\mu)^{-1}$$

for measures μ supported on the set E (subject to a regularity condition), where the s-energy is defined by

$$I_s(\mu) = \iint \frac{d\mu(x)\, d\mu(y)}{|x-y|^s}$$

for $s \geq 0$; this can be extrapolated from [30]. Although the Favard length is defined using the family of orthogonal projections $\{\pi_\theta : \theta \in [0, 2\pi]\}$, this can be substantially generalized. Other families of projection-like maps, such as radial projections and curve-adapted projections, can be used to define Favard-like averages related to the fine structure of a set. In [17], we found a *transversality* condition that many projection-type maps have in common; morally, a family of projection-type maps satisfies a transversality condition given strong enough separation; the views of a set from different perspectives are differentiated from each other. This turns out to be the key ingredient in linking energy to Favard estimates and connecting potentials to the geometric arrangement of the underlying set.

2.3. Current work: Minimal-energy measures on fractal sets.

Given the connection between Favard lengths and energies of measures, it is natural to study the structure of a (probability) measure which *minimizes* the s-energy on a set $E \subset \mathbb{C}$. One can approximate the energy-minimizing measure by taking a sum of area measures localized to the components of E. Rather than integrating $|x-y|^{-s}$, we estimate $\mathrm{dist}(Q,R)^{-1}$ for different components Q, R of E; this leads to a natural discretization

$$\min_{\substack{\mu(E)=1 \\ \mathrm{supp}\,\mu \subseteq E}} \iint |x-y|^{-s} d\mu(x)\, d\mu(y) \longrightarrow \min_{\mu(E)=1} \sum_{Q,R \text{ components of } E} \sum c_{Q,R}\, \mu(Q)\mu(R).$$

This double sum has the structure of a quadratic form $\vec{\mu}^T A \vec{\mu}$ where the matrix A incorporates the size, geometry, and arrangement of E and is always positive definite. After this translation, one can use numerical techniques to explicitly estimate the minimum energy of a measure supported on a given set and search for sets which have exceptionally large or small energy minimizers. Much of this work is numerical in nature, and many interesting examples arise from sets E which approximate well-known fractal sets. The connections between computation and geometry would make this project potentially accessible for undergraduate research.

This ongoing work is joint with Krystal Taylor.

2.4. **Future work.** Sharp understanding of the Favard length decay of an unrectifiable set E in the plane is conjectured to be connected with its analytic capacity $\gamma(E)$ (see, e.g. [27] and [31]); this is closely related with the Painlevé problem of finding a purely geometric characterization of removable sets for bounded holomorphic functions in the plane (see Tolsa's Salem Prize-winning work, e.g. [38]). As a model case, consider the four-corner Cantor set $\mathcal{K}$ in the plane (also called the Garnett Cantor set [23]); its generations are denoted $\mathcal{K}_n$, and consist of 4^n squares of side-length 4^{-n}. It is a simple case of a purely unrectifiable set with positive and finite Hausdorff measure at dimension 1. We have the following question:

Question. Does there exist a constant C such that $\text{Fav}(\mathcal{K}_n) \leq C\gamma(\mathcal{K}_n)$?

As a more general result, we hope that the Favard length is controlled by the analytic capacity for a more general class of sets, which would have powerful consequences. The analytic capacity was estimated precisely by Tolsa in [38] and is on the order $n^{-1/2}$ using curvature results; on the other hand, the Favard length is known to satisfy $\frac{\ln n}{n} \lesssim \text{Fav}(\mathcal{K}_n) \lesssim n^{-1/6+\delta}$ due to results of Bateman, Nazarov, Peres and Volberg ([6], [33]). Tightening this gap sufficiently would address the question.

In the case of the four-corner Cantor set, several geometric properties lead to an improved understanding of the Favard length. Since this set is so highly self-similar, my work on the convexity of the sequence of Favard lengths is applicable to improving the lower bounds. It is possible that improving the losses of this method will yield new lower bounds. Futhermore, the Cartesian product structure of this set can be exploited, bringing in Fourier analytic (as in [33] and [11]) and number theoretic techniques, as well as relationships with Falconer-style distance problems.

3. Commutators, BMO, and weighted inequalities

Given a symbol b and an operator T, the commutator $[b, T]$ can be formally defined by $[b, T]f = bT(f) - T(bf)$ for functions f in an appropriate space. When T is a singular integral operator (such as a Hilbert or Riesz transform) with a kernel K, this can give rise to a representation

$$[b, T]f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))K(x, y)\, dy.$$

Cancellation in the function b can then be paired with cancellation in the kernel K to give bounds for commutators on function spaces; it is then natural to study these operators for functions in the space of bounded mean oscillation (BMO). A classical result of Coifman, Rochberg, and Weiss [21] shows that the BMO norm of the symbol b can be completely characterized through the L^2 bounds for commutators against Riesz transforms.

3.1. **Commutators in a non-isotropic setting.** In a setting beyond a Calderón-Zygmund kernel, the characterization of L^2 bounds becomes much more difficult. One such operator, which arises naturally in curvature problems [37], is the *parabolic Hilbert transform*

$$H_\gamma f(x) = p.v. \int_{-\infty}^{\infty} f(x - \gamma(t))\, \frac{dt}{t}.$$

The singularity involved in this operator is localized to a curve, and induces much more complicated geometric structure and precludes classical techniques such as the median method of [26]. In [16], we gave a sufficient condition for L^2 bounds of commutators $[b, H_\gamma]$. There is a natural non-isotropic BMO space adapted to the family of parabolic cubes, which follows from a sparse domination result in [19]; the sparse theorem can then be paired with the weighted inequalities of [8] to give the desired upper bound. We also gave a necessary condition in terms of a new BMO-type space that is naturally adapted to the curve singularity.

Our techiques generalize naturally to a number of other contexts; for example, the Hilbert transform can be defined along a monomial curve or other curves satisfying a non-degeneracy condition.

3.2. **Current work: Harmonic function techniques and trace theorems.** Frequently, boundedness of commutators can be studied via duality: we estimate integrals of the form $\langle [b,T]f,g\rangle$ by pairing from an appropriate pair dual spaces such as L^p and $L^{p'}$. The work of Lenzmann and Schikorra [29] gave a useful trace inequality that represents this inner product as a boundary integral involving harmonic extensions. In essence, when φ is a function and R_j is the j-th Riesz transform, we have estimates of the form

$$|\langle [R_j,\varphi]f,g\rangle| \lesssim \int_{\mathbb{R}^{n+1}_+} |t\nabla\Phi(x,t)|\cdot|F(x,t)|\cdot|t\nabla G(x,t)|\frac{dx\,dt}{t},$$

where capital letters indicate the harmonic extensions of functions on $\mathbb{R}^n$ into the upper half space $\mathbb{R}^{n+1}_+$. In the classical, unweighted setting it is possible to estimate this integral using tent-cone duality and standard square function estimates to bound the integral by the product of L^2-norms of f and g and the BMO norm of φ. This yields the key result: the commutator $[R_j,\varphi]$ is bounded whenever φ has bounded mean oscillation.

In the two-weight setting, where $f\in L^2(\mu)$ and $g\in L^2(\lambda^{-1})$ lie in weighted L^2 spaces with weights μ,λ satisfying Muckenhoupt's A_2 condition, things are far more delicate; in particular, the BMO norm is no longer the right tool to characterize the symbols for which $[R_j,\varphi]$ is bounded and must be replaced by a weight-adapted BMO norm. Using weight-adapted square function estimates, it is possible to show that

$$|\langle [R_j,\varphi]f,g\rangle| \lesssim \|f\|_{L^2(\mu)}\|g\|_{L^2(\lambda^{-1})}\cdot\left(\sup_B \frac{1}{\mu(B)}\iint_{T(B)} |t\nabla\Phi(y,t)|^2\frac{d\lambda(y)\,dt}{t}\right)^{1/2}$$

where the supremum is taken over balls $B\subseteq\mathbb{R}^n$ and $T(B)$ is the associated tent in $\mathbb{R}^{n+1}_+$ over the ball. Serious obstacles arise when trying to estimate this latter integral; the key problem is that the decay of the singular kernel underlying $|t\nabla\Phi|$ is most useful when the measure $d\lambda$ has doubling which is substantially better than most A_2 weights and we are working to remove the stronger doubling assumptions.

This ongoing work is joint with Brett Wick and Marie-Jose Kuffner.

3.3. **Future work.** A frequent approach to commutator inequalities is to use the Cauchy integral trick originated in [21], which translates commutator bounds into weighted bounds; estimates can then be derived from the classical connection between BMO and the space of A_2 weights. A natural generalization of this problem is to extend this to weighted bounds for commutators in either the one- or two-weight setting; this leads to

Question. For a given singular integral operator T, characterize the weights μ and λ and symbols b for which $[b,T]: L^2(\mu)\to L^2(\lambda)$ is bounded.

There are a number of approaches to this question for particular cases; for example, a one-weight characterization can be found with the A_2 absorption inequalities of [7]. A characterization for the Riesz transforms in terms of a weight-adapted BMO space (originally found in [10]) was given by [25] via paraproducts. We are working to find more direct approaches to one- and two-weight bounds for a large class of operators, such as monomial Hilbert transforms or the Ornstein-Uhlenbeck operator.

4. References

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